MODULAR SYSTEM

INTEGRALS

Ahmet ÇAKIR

Zambak PUBLISHING

http://book.zambak.com

Digital Assembly
Zambak Typesetting & Design

Page Design
Şamil Keskinoğlu

Proofreader
Zoe Barnett

Publisher
Zambak Basım Yayın Eğitim ve Turizm İşletmeleri Sanayi Ticaret A.Ş.

Printed by
Çağlayan A.Ş. Sarnıç Yolu Üzeri No:7
Gaziemir / Izmir, March 2008
Tel: +90-0-232-252 22 85
+90-0-232-522-20-96-97

ISBN: 978-975-266-222-3

Printed in Turkey

DISTRIBUTION
ZAMBAK YAYINLARI
Bulgurlu Mah. Haminne Çeşmesi Sok.
No. 20 34696 Üsküdar / Istanbul

Tel.: +90-216 522 09 00 (pbx)
Fax: +90-216 443 98 39
http://book.zambak.com

PREFACE

To the teacher

This book is about indefinite integrals, definite integrals, and integral applications. It is divided into three chapters, structured as follows:

- *Chapter 1 covers indefinite integrals. The first section of this chapter provides a basic introduction to integrals and integration rules. The second section looks at the main methods for evaluating integrals, including integration by substitution, integration by parts, and the integration of partial fractions.*
- *Chapter 2 covers definite integrals. The first section looks at the concept of the definite integral and its properties, and introduces two important theorems: the Fundamental Thorem of Calculus and the Mean Value Theorem. The second section is optional, and covers the integration of three particular types of function: absolute value functions, sign functions, and floor functions.*
- *Chapter 3 builds on the material of the previous chapter by showing some practical applications of the definite integral: finding the area under a curve and the length of a curve, and calculating the volume and surface area of a solid of revolution.*

Key emphasis is placed on the methods of integration by substitution and integration by parts in the second half of Chapter 1 and throughout the rest of the book. I consider these methods to be the most important ones, as they form the basis of many of the other integration techniques. Therefore, if the student masters these two methods, he or she will be well equipped to approach any integration problem.

The book follows a step-by-step teaching approach, which leads the student from basic definitions and concepts to a gradual mastery of the topic, through a large number of clear, solved examples. At each stage, students' progress can be checked through regular 'Check Yourself' sections and graded exercises at the end of each section (see the section 'Using this Book' at the end of this preface for more information). In addition, the Chapter Review Tests at the end of each chapter check students' understanding of the concepts and techniques in the whole chapter. The review tests are graded from easy to hard, with review test A being the easiest, review test B being slightly harder, and review test C (if present) being the hardest level of all.

To the student

Integration is important in many areas of mathematics, engineering and architecture. If you are planning to study any of these things beyond high school, you will need a good, basic understanding of integrals before you begin your university course. This book has been written to help you understand integrals when you study them for the first time. Take time to understand the material in this book, as your university teachers will expect you to know it when you arrive at university.

Integration is generally the last topic you will cover in your high school math course. The reason for this is simple: integration builds on a lot of the math you have covered in previous classes. Therefore, before you begin, make sure you have a good understanding of the following topics: polynomials, rational expressions, trigonometry, and logarithms.

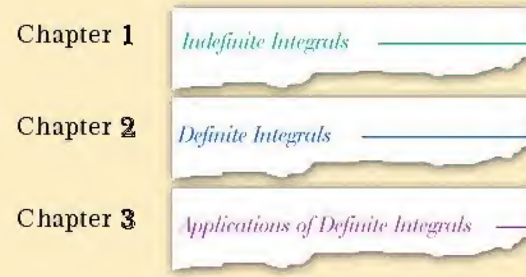
Acknowledgements

Many colleagues gave me invaluable help and advice during the writing of this book. I would like to thank everybody who helped me at Zambak Publications, especially Mustafa Kırıkçı, Cem Giray, and Şükrü Kavlu. Their encouragement and constructive comments were invaluable to me. Special thanks also go to Şamil Keskinoğlu for his patient typesetting and design, and to Zoe Barnett for her careful proofreading.

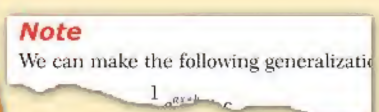
Finally, I would like to thank my family for their patience while I was working on this book.

To the Student: Using This Book

This book is designed so that you can use it effectively. Each chapter has its own special color that you can see at the bottom of the page.

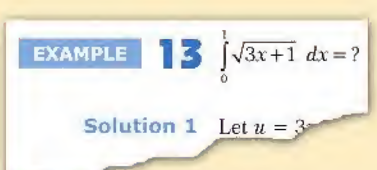


Different pieces of information in this book are useful in different ways. Look at the types of information and how they appear in the book:



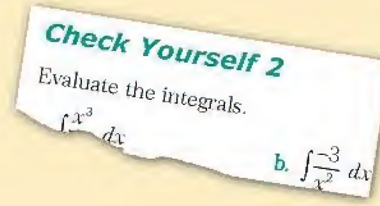
Notes help you focus on important details. When you see a note, read it twice! Make sure you understand it.

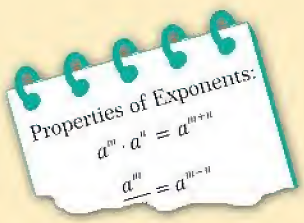
Definition boxes give formal descriptions of new concepts. Rule boxes give direct methods for finding the answers to questions. Theorem boxes include propositions that can be proved. The information in these boxes is very important for further understanding and for solving examples.



Examples include problems related to the topic and their solution, with explanations. The examples are numbered, so you can find them easily in the book.

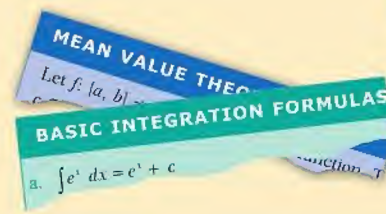
Check Yourself sections help you check your understanding of what you have just studied. Solve the problems alone and then check your answers against the answer key provided. If your answers are correct, you can move on to the next section. If your answer is wrong, go through your working again and check back through the examples in the section.



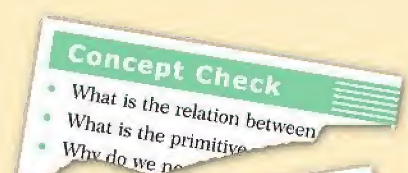


A small notebook in the left margin of a page reminds you of material that is related to the topic you are studying. Notebook text helps you to remember the math you need to understand the material. It might help you to see your mistakes, too! Notebooks are the same color as the section you are studying.

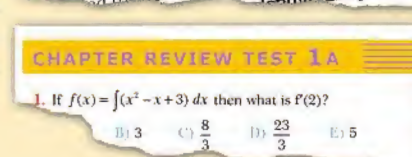
Special windows highlight important new information. Windows may contain formulas, properties, or solution procedures, etc. They are the same color as the color of the section.



Exercises at the end of each section cover the material in the whole section. You should be able to solve all the problems which do not have a star. One star (✪) next to a question means the question is a bit harder. Two stars (✪✪) next to a question mean the question is for students who are looking for a challenge! The answers to the exercises are at the back of the book.



The Chapter Summary at the end of each chapter summarizes all the important material that has been covered in the chapter. The Concept Check section contains oral questions. In order to answer them you don't need paper or pen. If you answer Concept Check questions correctly, it means you know that topic! The answers to these questions are in the material you studied. Go back over the material if you are not sure about an answer to a Concept Check question. Finally, the Chapter Review tests include questions in increasing order of difficulty and contain multiple choice questions. The answer key for these tests is at the back of the book.



CONTENTS

INTEGRALS

CHAPTER 1: INDEFINITE INTEGRALS

1. ANTIDERIVATIVE AND INDEFINITE INTEGRAL 2
   - A. DEFINITION OF THE INDEFINITE INTEGRAL 2
   - B. PROPERTIES OF THE INDEFINITE INTEGRAL 4
   - C. BASIC INTEGRATION FORMULAS 6

EXERCISES 1.1 16

2. INTEGRATION METHODS 18
   - A. INTEGRATION BY SUBSTITUTION 18
   - B. INTEGRATION BY PARTS 25
   - C. INTEGRATING PARTIAL FRACTIONS 30
   - D. INTEGRATING RADICAL FRACTIONS 37
   - E. INTEGRATING TRIGONOMETRIC FUNCTIONS 41

EXERCISES 1.2 47

CHAPTER SUMMARY 49

CONCEPT CHECK 50

CHAPTER REVIEW TEST 1A 51

CHAPTER REVIEW TEST 1B 53

CHAPTER REVIEW TEST 1C 55

CHAPTER 2: DEFINITE INTEGRALS

1. EVALUATING DEFINITE INTEGRALS 58
   - A. DEFINITION OF THE DEFINITE INTEGRAL 58
   - B. THE FUNDAMENTAL THEOREM OF CALCULUS 60
   - C. PROPERTIES OF THE DEFINITE INTEGRAL 61
   - D. LEIBNIZ'S RULE 66
   - E. THE MEAN VALUE THEOREM 67

EXERCISES 2.1 68

2. INTEGRALS OF SOME SPECIAL FUNCTIONS (OPTIONAL) 69
   - A. INTEGRATING ABSOLUTE VALUE FUNCTIONS 69
   - B. INTEGRATING SIGN FUNCTIONS 71
   - C. INTEGRATING FLOOR FUNCTIONS 72

EXERCISES 2.2 75

CHAPTER SUMMARY 76

CONCEPT CHECK 76

CHAPTER REVIEW TEST 2A 77

CHAPTER REVIEW TEST 2B 79

CHAPTER 3: APPLICATIONS OF DEFINITE INTEGRALS

1. FINDING THE AREA UNDER A CURVE 82

EXERCISES 3.1 89

2. OTHER APPLICATIONS 92
   - A. CALCULATING THE VOLUME OF A SOLID OF REVOLUTION 92
   - B. FINDING THE LENGTH OF A CURVE (*OPTIONAL*) 96
   - C. CALCULATING THE AREA OF A SURFACE OF REVOLUTION (*OPTIONAL*) 97

EXERCISES 3.2 99

CHAPTER SUMMARY 101

CONCEPT CHECK 101

CHAPTER REVIEW TEST 3A 102

CHAPTER REVIEW TEST 3B 104

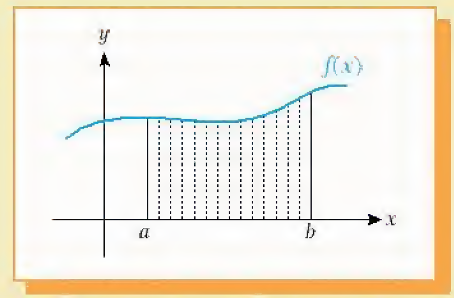
ANSWERS 106

GLOSSARY 110

INTRODUCTION

From your previous studies you know how to find the derivative of a function. By using the derivative we can find the slope of a tangent line to a function at a point, as well as the intervals of monotony and local maximum and minimum points of functions.

In this book we have one question to answer: we can find the derivative of a function, but if we are given the derivative of a function then can we find the original function? Trying to answer this question leads us to the concept of integration.



We can define integration in two ways:

1. as the process of finding the primitive function of a given derivative, or

2. as the process of finding the infinite sum of small parts.

This gives us two types of integral. The first type of integral is called the indefinite integral (or antiderivative), and the second type is called the definite integral.

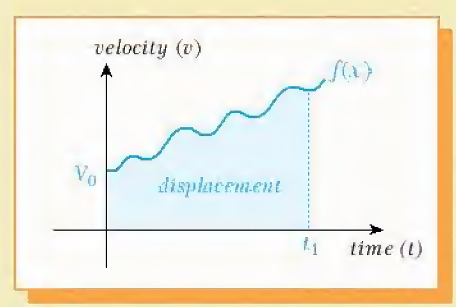
The History of Integrals

We can say that integration began with the problem of finding the area between a quadratic or cubic function and one or more axes. Another related problem was finding the volume of revolution that we generate by revolving a figure around any axis. Mathematicians began studying these problems in ancient times.

Hippocrates (440 B.C.) tried to find the area under a curve by dividing the curve into some small rectangles. This gave an approximate value of the area under a curve.

After Hippocrates, Archimedes (287-212 B.C.) used a polygon with 96 sides to find the area under a curve and also the area of a circle, and calculated the number π approximately. Archimedes tried to divide the area under a curve into hundreds of infinitesimal rectangles to obtain his results.

Muslim mathematicians also tried to find the area under a curve. Thabit ibn Qurrah (826-901) found the area of a circle using complicated methods. In the eleventh century, Ibn al-Haytham evaluated the volume of revolution around any axis.



In the fourteenth century, Heytesbury found the formula for calculating the distance of an object moving at a uniform velocity. The area under a velocity-time graph gives the distance, and the area under an acceleration-time graph gives the velocity of the object. By using this information, Nicole Oresme invented kinematics.

In the fifteenth and sixteenth centuries, European sailors were exploring the world and needed to find the distance between any two points on the Earth's surface. For this reason, Edward Wright evaluated the approximate value of the integral of the secant function. In the seventeenth century, Kepler evaluated the volume of many types of solids of revolution.

Fermat (1601-1665) evaluated the area under the curve of $y = ax^n$ by using an infinitesimal number of inscribed and circumscribed rectangles. At the same time, Vincent evaluated the integral of $y=\frac{1}{x}$ and his student Sasara found the other logarithmic functions, then Neile calculated the arc length of a curve.

In 1670 Bartow collected all known information about integrals in a book. One of his followers, Newton (1642-1727), studied these topics and wrote a book called *On the Quadrature of Curves* about integrals. In his work he developed the Fundamental Theorem of Calculus. By using this theorem he found many of the formulas of integration, including the substitution method and integration by parts.

Gottfried Wilhelm Leibniz (1646-1716) thought about the area under a curve and divided it into infinitely infinitesimal rectangles, whose infinite sum gives us the total area. In his calculations he used the symbol $\int y\,dx$. In this notation, $\int$ is the first letter of Greek word *summa* (meaning 'sum'), y is the ordinate of a given point, and dx is the differential (i.e. dx is the width of the rectangle and y is the height. If we multiply them we find the area of a small part. If we add all these infinitely many parts we obtain the total area.)

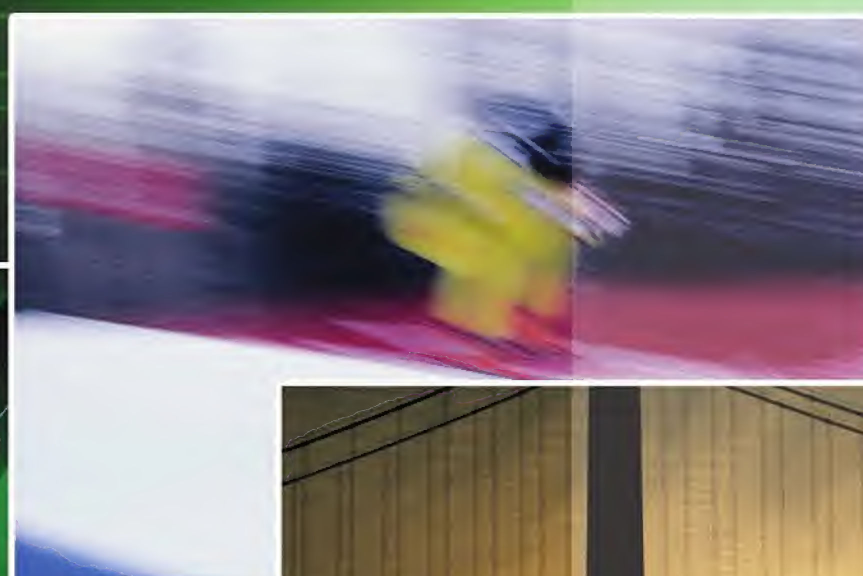
Newton's Fundamental Theorem of Calculus tells us that the integral is the inverse of the derivative: it is the antiderivative. However, Leibniz's discoveries and integral notation were also important for our understanding of the integral as the area under a curve. For this reason, we can say that Newton and Leibniz are both fathers of the integral.

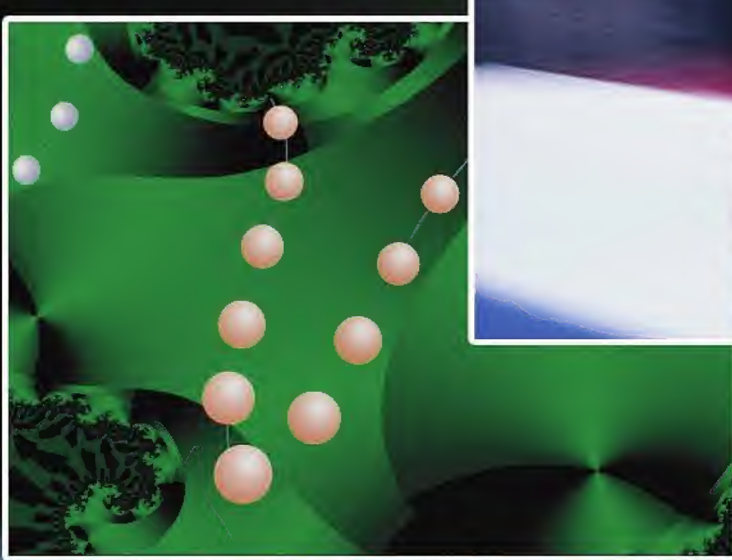
Other important mathematicians in the development of integral calculus are Bernoulli (who studied the integration of partial fractions), Cauchy (for his work on integrals as the limit of infinite sums, and the Mean Value Theorem), and Riemann (who developed general formulas for the integral of any function).

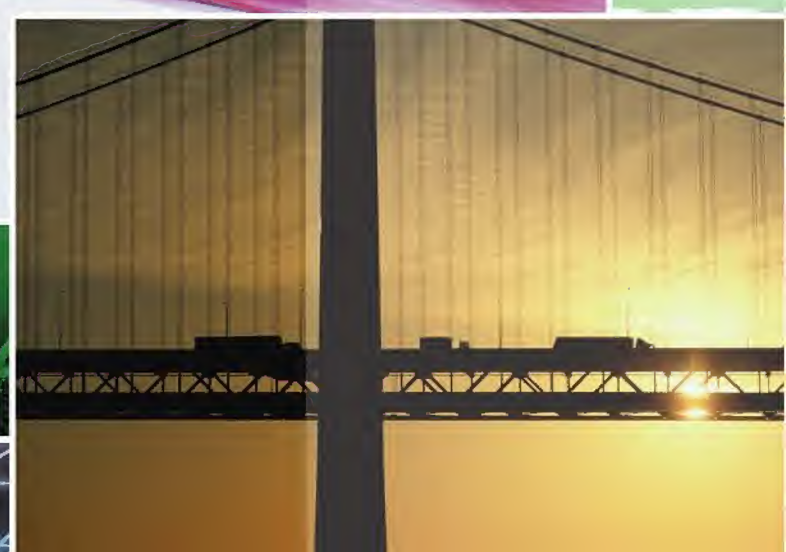
Mathematicians, physicists, architects and engineers today use many advanced properties of integrals and special integration methods. In this book, we will begin our study of the subject by looking at the basic rules and definitions developed by Newton, Leibniz and their successors.

Chapter 1

INDEFINITE INTEGRALS







1 ANTIDERIVATIVE AND INDEFINITE INTEGRAL

A. DEFINITION OF THE INDEFINITE INTEGRAL

The set of all antiderivatives of a function is called the indefinite integral of the given function.

Definition

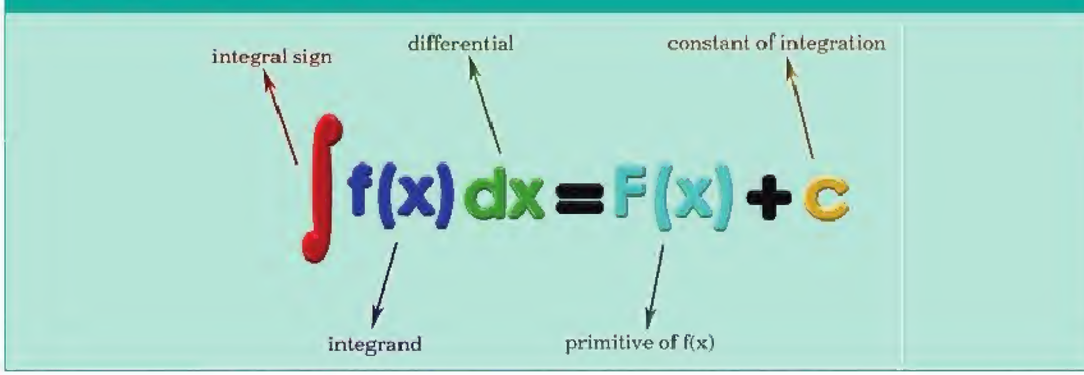
indefinite integral

Let $F(x)$ be a differentiable function such that $F'(x) = f(x)$. Then $F(x)$ is called the primitive or antiderivative of the function $f(x)$, and the expression $F(x) + c$ is called the indefinite integral of $f(x)$.

We write the indefinite integral as follows: $\int f(x)\,dx = F(x) + c$.

The different parts of the expression are as follows:

THE INDEFINITE INTEGRAL



The differential dx in the expression shows that we mean the indefinite integral with respect to the variable x. We can also find the indefinite integral with respect to other variables, for example:

$$\int f(t)\,dt = F(t) + c.$$

We read the expression $\int f(x)\,dx$ as 'the integral of $f(x)$ with respect to x'. The process of finding the integral of a function is called integration.

To find the integral of $f(x)$ we ask the question: 'the derivative of which function is $f(x)$?'.

In other words, integration is the reverse operation of differentiation. This is why the integral is sometimes called the antiderivative of a function, plus a constant term. Let us look at some examples.

EXAMPLE 1 $\int 2x\ dx = x^2 + c$

(The derivative of x^2 is $2x$, so $\int 2x\ d(x) = x^2 + c$.)

EXAMPLE 2 $\int 4x^3\ dx = x^4 + c$

(The derivative of x^4 is $4x^3$, so $\int 4x^3\ dx = x^4 + c$.)

EXAMPLE 3 $\int \sin x\ dx = -\cos x + c$

(The derivative of $-\cos x$ is $\sin x$, so $\int \sin x\ dx = -\cos x + c$.)

In examples 1 to 3 we add a constant c to each primitive. Why do we use it? To understand the reason, let us look at the derivatives of three different functions:

$$y = x^2, \qquad y' = 2x;$$
$$y = x^2 + 3, \qquad y' = 2x;$$
$$y = x^2 - 6, \qquad y' = 2x.$$

We can see that the derivatives of the three functions are the same but the primitive functions are different. In other words, the integral of $2x$ could be x^2, or $x^2 + 3$, or $x^2 - 6$, or any other expression of the form $x^2 + c$, where c is a constant number (i.e. not a variable). Similarly, the integral of $3x^2$ could be $x^3 + 7$, or $x^3 - 12$, or any expression of the form $x^3 + c$. In fact, when we find the indefinite integral of any function we need to add a constant term, c. This constant is called the constant of integration.

Note

We must always use the constant of integration when finding an indefinite integral.

EXAMPLE 4 If $f'(x) = 2x$ and $f(3) = 7$ then find $f(x)$.

Solution $f(x) = \int f'(x)\ dx = \int 2x\ dx = x^2 + c$

$f(3) = 3^2 + c = 7$

$c = -2$

So $f(x) = x^2 - 2$.

EXAMPLE 5 $\int dx = ?$

Solution We know that $dx = 1 \cdot dx$ and $y = x$, $y' = 1$.

So $\int dx = \int 1 \cdot dx = x + c$.

EXAMPLE 6 $\int dy = ?$

Solution This is the same as Example 5 except we need to integrate with respect to y. So $\int dy = y + c$.

EXAMPLE 7 $\int d(\tan x) = ?$

Solution Using $\tan x$ as variable, $\int d(\tan x) = \int 1 \cdot d(\tan x) = \tan x + c$.

B. PROPERTIES OF THE INDEFINITE INTEGRAL

1. The differential of the indefinite integral is equal to the expression after the integral sign:
$$d\int f(x)\,dx = f(x)\,dx.$$
2. The derivative of the indefinite integral is equal to the integrand:
$$\frac{d}{dx}\int f(x)\,dx = \int \frac{d}{dx} f(x)\,dx = f(x).$$
The integral and derivative are inverse operations so they simplify each other.
3. The indefinite integral of the differential of a function is the same function with a constant term added:
$$\int dF(x) = F(x) + c.$$
4. Constant multipliers can be taken outside of the integral sign:
$$\int a \cdot f(x)\,dx = a \cdot \int f(x)\,dx.$$
5. The integral of the sum or difference of two functions is equal to the sum or difference of the integrals of the given functions:
$$\int [f(x) \pm g(x)]\,dx = \int f(x)\,dx \pm \int g(x)\,dx.$$

Proof To prove each property we will use the definition of integration:

if $F'(x) = f(x)$ then $\int f(x)\,dx = F(x) + c$.

1. Differentiating both sides of the equation $\int f(x)\,dx = F(x) + c$ gives:
$d\int f(x)\,dx = d(F(x) + c) = F'(x)\,dx + 0\,dx = f(x)\,dx.$
So $d\int f(x)\,dx = f(x)\,dx.$
2. Again, using the definition we get
$\frac{d}{dx}\int f(x)\,dx = (\int f(x)\,dx)' = (F(x) + c)' = F'(x) + c' = f(x) + 0 = f(x).$
The proofs of properties 3 and 4 are left as an exercise for you.
5. Let us take the derivative of both sides of the statement to be proved:
$(\int [f(x) \pm g(x)]\,dx)' = (\int f(x)\,dx \pm \int g(x)\,dx)'$
$f(x) \pm g(x) = (\int f(x)\,dx)' \pm (\int g(x)\,dx)' = f(x) \pm g(x).$

EXAMPLE 8 $f(x) = \int d(x^3 - 2x^2 + 1)$ is given. Find $f(2)$ if $f(1) = 0$.

Solution Let us use the third property of integration:

$$f(x) = \int d(x^3 - 2x^2 + 1) = x^3 - 2x^2 + 1 + c$$

$$f(1) = 1^3 - 2 \cdot 1^2 + 1 + c = 0 \Rightarrow c = 0 \text{ and } f(x) = x^3 - 2x^2 + 1$$

$$f(2) = 2^3 - 2 \cdot 2^2 + 1 = 8 - 8 + 1 = 1.$$

EXAMPLE 9 $\int x^2 \cdot f(x)\, dx = 3x^4 + 4x^3 - x^2$ is given. Find $f(x)$. $(x \neq 0)$

Solution We can use the second property of integration. Take the derivative of both sides:

$$\frac{d}{dx}\int x^2 \cdot f(x)\, dx = \frac{d}{dx}(3x^4 + 4x^3 - x^2)$$

$$x^2 \cdot f(x) = 12x^3 + 12x^2 - 2x$$

$$f(x) = 12x + 12 - \frac{2}{x}.$$

EXAMPLE 10 Evaluate the integral $\int (\sin x - 3x^2 + e^x)\, dx$.

Solution By the fifth property of integration, $\int (\sin x - 3x^2 + e^x)\, dx = \int \sin x\, dx - \int 3x^2\, dx + \int e^x\, dx$.

But we know that $(\cos x)' = -\sin x$, $(x^3)' = 3x^2$ and $(e^x)' = e^x$,

so $\int \sin x\, dx - \int 3x^2\, dx + \int e^x\, dx = -\cos x - x^3 + e^x + c$.

EXAMPLE 11 $f'(x) = 12x^3 + 6x^2 - 4x - 5$ and $f(2) = 5$ are given. Find the value of $f(1)$.

Solution We can use the second, fourth and fifth properties of integration. Integrating both sides gives:

$$\int f'(x)\, dx = \int (12x^3 + 6x^2 - 4x - 5)\, dx$$

$$f(x) = \int 12x^3\, dx + \int 6x^2\, dx - \int 4x\, dx - \int 5\, dx$$

$$= 3 \cdot \int 4x^3\, dx + 2 \cdot \int 3x^2\, dx - 2\int 2x\, dx - 5\int dx$$

$$= 3 \cdot x^4 + c_1 + 2 \cdot x^3 + c_2 - 2 \cdot x^2 + c_3 - 5x + c_4$$

$$= 3x^4 + 2x^3 - 2x^2 - 5x + (c_1 + c_2 + c_3 + c_4).$$

However, the expression $c_1 + c_2 + c_3 + c_4$ is equal to any constant real number, so we can replace it with the single constant c:

$f(x) = 3x^4 + 2x^3 - 2x^2 - 5x + c$.

Now $f(2) = 5$ (given), so $5 = 3 \cdot 2^4 + 2 \cdot 2^3 - 2 \cdot 2^2 - 5 \cdot 2 + c = 48 + 16 - 8 - 10 + c = 46 + c$.

So $c = -41$ and $f(x) = 3x^4 + 2x^3 - 2x^2 - 5x - 41$, which means

$f(1) = 3 \cdot 1^4 + 2 \cdot 1^3 - 2 \cdot 1^2 - 5 \cdot 1 - 41 = -43$.

Check Yourself 1

1. Evaluate each integral by using the definition of integration.

   a. $\int 6x^5\, dx$ b. $\int 3\sin x\, dx$ c. $\int 4e^{2x}\, dx$

2. Use the properties of the indefinite integral to calculate each integral.

   a. $\int (x^3 + 4x^2 - 3x - 1)\, dx$ b. $\int \left(\frac{1}{x^3} + 4\cos x - e^x\right) dx$

3. $\int x^3 \cdot f(x)\, dx = x^5 - 4x^3 + 2x^2 + 1$ is given. Find $f(x)$.

Answers

1. a. $x^6 + c$ b. $-3\cos x + c$ c. $2e^{2x} + c$

2. a. $\frac{x^4}{4} + \frac{4x^3}{3} - \frac{3x^2}{2} - x + c$ b. $-\frac{1}{2x^2} + 4\sin x - e^x + c$ 3. $f(x) = 5x - \frac{12}{x} + \frac{4}{x^2}$

C. BASIC INTEGRATION FORMULAS

We have seen that integration is the opposite of differentiation. Therefore we can take the formulas we found for the derivative and 'reverse' them to obtain formulas for the integral of a function. Let us look at each set of formulas in turn, along with some examples of their application.

Note

The formulas in this section come from 'reversing' the formulas we found for the derivative. We will not prove them here.

BASIC INTEGRATION FORMULAS - 1

a. $\int x^n\, dx = \frac{x^{n+1}}{n+1} + c \qquad n \neq -1$

b. $\int a\, dx = ax + c$ for $a \in R$

EXAMPLE 12 $\int x^3\ dx = ?$

Solution Using formula 1a: $\int x^3\ dx = \dfrac{x^{3+1}}{3+1} + c = \dfrac{x^4}{4} + c.$

EXAMPLE 13 $\int \dfrac{1}{x^4}\ dx = ?$

Solution $\int \dfrac{1}{x^4}\ dx = \int x^{-4}\ dx = \dfrac{x^{-4+1}}{-4+1} + c = \dfrac{x^{-3}}{-3} + c = -\dfrac{1}{3x^3} + c$

EXAMPLE 14 $\int 3x^5\ dx = ?$

Solution $\int 3x^5\ dx = 3\int x^5\ dx = 3 \cdot \dfrac{x^6}{6} + c = \dfrac{x^6}{2} + c$

EXAMPLE 15 $\int (3x^4 + 4x^3 - 2x^2 + x - 5)\ dx = ?$

Solution $\int (3x^4 + 4x^3 - 2x^2 + x - 5)\ dx = 3\dfrac{x^5}{5} + 4\dfrac{x^4}{4} - 2\dfrac{x^3}{3} + \dfrac{x^2}{2} - 5x + c$

$$= \frac{3x^5}{5} + x^4 - \frac{2x^3}{3} + \frac{x^2}{2} - 5x + c$$

Check Yourself 2

Evaluate the integrals.

a. $\int \dfrac{x^3}{2}\ dx$ b. $\int \dfrac{-3}{x^2}\ dx$ c. $\int (x^2 - x^3 + x^4)\, dx$

d. $\int (x-1)\sqrt{x}\ dx$ e. $\int (x+1)^2\ dx$ f. $\int (3x^3 + 4x^2 - x)\, dx$

Answers

a. $\dfrac{x^4}{8} + c$ b. $\dfrac{3}{x} + c$ c. $\dfrac{x^3}{3} - \dfrac{x^4}{4} + \dfrac{x^5}{5} + c$

d. $\dfrac{2x^{5/2}}{5} - \dfrac{2x^{3/2}}{3} + c$ e. $\dfrac{x^3}{3} + x^2 + x + c$ f. $\dfrac{3x^4}{4} + \dfrac{4x^3}{3} - \dfrac{x^2}{2} + c$

BASIC INTEGRATION FORMULAS - 2

a. $\int \frac{1}{x}\, dx = \ln|x| + c$

b. $\int \frac{u'(x)\, dx}{u(x)} = \ln|u(x)| + c$

EXAMPLE 16 $\int \frac{1}{y}\, dy = ?$

Solution $\int \frac{1}{y}\, dy = \ln|y| + c$

EXAMPLE 17 $\int \frac{2x}{x^2}\, dx = ?$

Solution Using formula 2b: $\int \frac{2x}{x^2}\, dx = \ln\left|x^2\right| + c = \ln x^2 + c$ since $(x^2)' = 2x$

EXAMPLE 18 $\int \frac{\cos x}{\sin x}\, dx = ?$

Solution $\int \frac{\cos x}{\sin x}\, dx = \ln|\sin x| + c$ since $(\sin x)' = \cos x$

EXAMPLE 19 $\int \frac{4}{x-3}\, dx = ?$

Solution $\int \frac{4}{x-3}\, dx = 4 \cdot \ln|x-3| + c$ since $(x-3)' = 1$

Check Yourself 3

Evaluate the integrals.

a. $\int \left(\frac{1}{x} - \frac{1}{x^2} - \frac{1}{x^3}\right) dx$ b. $\int \frac{5x^5 + 2x^2 + 3x - 5}{x^2}\, dx$ c. $\int \frac{3}{1+5x}\, dx$

Answers

a. $\ln|x| + \frac{1}{x} + \frac{1}{2x^2} + c$ b. $\frac{5x^4}{4} + 2x + 3\ln|x| + \frac{5}{x} + c$ c. $\frac{3}{5}\ln|5x+1| + c$

BASIC INTEGRATION FORMULAS - 3

a. $\int e^x \, dx = e^x + c$

b. $\int a^x \, dx = \dfrac{a^x}{\ln a} + c$

EXAMPLE 20 $\int 3^x \, dx = ?$

Solution $\int 3^x \, dx = \dfrac{3^x}{\ln 3} + c$

EXAMPLE 21 $\int 4 \cdot 5^x \, dx = ?$

Solution $\int 4 \cdot 5^x \, dx = 4 \cdot \int 5^x \, dx = \dfrac{4 \cdot 5^x}{\ln 5} + c$

EXAMPLE 22 $\int 4e^x \, dx = ?$

Solution $\int 4e^x \, dx = 4e^x + c$

EXAMPLE 23 $\int 7^{4x-3} \, dx = ?$

Solution $\int 7^{4x-3} \, dx = \int \dfrac{(7^4)^x}{7^3} dx = \dfrac{(7^4)^x}{7^3 \cdot \ln 7^4} + c = \dfrac{7^{4x-3}}{4 \cdot \ln 7} + c$

Properties of Exponents:

$a^m \cdot a^n = a^{m+n}$

$\dfrac{a^m}{a^n} = a^{m-n}$

$(a^m)^n = a^{mn}$

EXAMPLE 24 $\int 5e^{x+1} \, dx = ?$

Solution $\int 5e^{x+1} \, dx = 5e \cdot \int e^x \, dx = 5e \cdot e^x + c = 5e^{x+1} + c$

EXAMPLE 25 $\int e^{3x+1}\,dx = ?$

Solution $\int e^{3x+1}\,dx = e\cdot\int (e^3)^x\,dx = e\cdot\dfrac{(e^3)^x}{\ln(e^3)} + c = \dfrac{e^{3x+1}}{3} + c.$

Note

We can make the following generalizations of formulas 3a and 3b:

$$\int e^{ax+b}\,dx = \frac{1}{a}e^{ax+b} + c, \qquad (1)$$

$$\int a^{mx+n}\,dx = \frac{1}{m}\frac{a^{mx+n}}{\ln a} + c \qquad (2)$$

EXAMPLE 26 $\int e^{4x-1}dx = ?$

Solution By (1) above, $\int e^{4x-1}\,dx = \dfrac{1}{4}e^{4x-1} + c.$

EXAMPLE 27 $\int 5^{3x+1}dx = ?$

Solution By (2) above,

$$\int 5^{3x+1}\,dx = \frac{1}{3}\frac{5^{3x+1}}{\ln 5} + c = \frac{5^{3x+1}}{3\ln 5} + c.$$

Check Yourself 4

Evaluate the integrals.

a. $\int 5\cdot 3^x\,dx$

b. $\int (2^x - 3^x)\,dx$

c. $\int 6^{5x+1}\,dx$

d. $\int \left(3\cdot 5^{x+3} + \dfrac{4^{x+3}}{3}\right)dx$

e. $\int 3e^{3x}\,dx$

f. $\int 2e^{(2x-3)}\,dx$

g. $\int 2^{4x-4}\,dx$

h. $\int 4^{ex+\pi}\,dx$

Answers

a. $\dfrac{5\cdot 3^x}{\ln 3} + c$

b. $\dfrac{2^x}{\ln 2} - \dfrac{3^x}{\ln 3} + c$

c. $\dfrac{6^{5x+1}}{5\ln 6} + c$

d. $\dfrac{3\cdot 5^{x+3}}{\ln 5} + \dfrac{4^{x+3}}{3\ln 4} + c$

e. $e^{3x} + c$

f. $e^{2x-3} + c$

g. $\dfrac{2^{4x-6}}{\ln 2} + c$

h. $\dfrac{2^{2ex+2\pi-1}}{e\cdot\ln 2} + c$

BASIC INTEGRATION FORMULAS - 3

a. $\int \sin x \; dx = -\cos x + c$

b. $\int \sin(ax+b) \; dx = -\frac{1}{a}\cos(ax+b) + c$

c. $\int \cos x \; dx = \sin x + c$

d. $\int \cos(ax+b) \; dx = \frac{1}{a}\sin(ax+b) + c$

e. $\int \frac{1}{\cos^2 x} \; dx = \int \sec^2 x \; dx = \int (1+\tan^2 x) \; dx = \tan x + c$

f. $\int \frac{1}{\cos^2(ax+b)} \; dx = \int \sec^2(ax+b) \; dx = \int (1+\tan^2(ax+b)) \; dx = \frac{1}{a}\tan(ax+b) + c$

g. $\int \frac{1}{\sin^2 x} dx = \int \csc^2 x dx = \int (1+\cot^2 x) dx = -\cot x + c$

h. $\int \frac{1}{\sin^2(ax+b)} \; dx = \int \csc^2(ax+b) \; dx = \int (1+\cot^2(ax+b)) \; dx = -\frac{1}{a}\cot(ax+b) + c$

i. $\int \frac{1}{\sqrt{1-x^2}} dx = \arcsin x + c_1 = -\arccos x + c_2$

j. $\int \frac{1}{1+x^2} dx = \arctan x + c_1 = -\text{arccot}\, x + c_2$

Inverse Trigonometric Identities:

$\arcsin x + \arccos x = \frac{\pi}{2}$

$\arctan x + \text{arccot}\, x = \frac{\pi}{2}$

EXAMPLE 28 $\int 4\sin x \; dx = ?$

Solution $\int 4\sin x \; dx = 4 \cdot \int \sin x \; dx = -4\cos x + c$

EXAMPLE 29 $\int \sin 3x \; dx = ?$

Solution $\int \sin 3x \; dx = -\frac{1}{3}\cos 3x + c$

EXAMPLE 30 $\int \cos(7x+3)\ dx = ?$

Solution $\int \cos(7x+3)\ dx = \frac{1}{7}\sin(7x+3) + c$

EXAMPLE 31 $\int 5\sin(2x-1)\ dx = ?$

Solution $\int 5\sin(2x-1)\ dx = 5\cdot\int \sin(2x-1)\ dx = 5\cdot\frac{1}{2}\cdot(-\cos(2x-1)) + c = -\frac{5}{2}\cos(2x-1) + c$

EXAMPLE 32 $\int \frac{1}{\cos^2 3x}\ dx = ?$

Solution $\int \frac{1}{\cos^2 3x}\ dx = \frac{1}{3}\cdot\tan 3x + c$

EXAMPLE 33 $\int \csc^2(5x-3)\ dx = ?$

Solution $\int \csc^2(5x-3)\ dx = -\frac{1}{5}\cdot\cot(5x-3) + c$

EXAMPLE 34 $\int \cos^2 x\ dx = ?$

Solution

$$\begin{aligned}
\int \cos^2 x\ dx &= \int \frac{1+\cos 2x}{2}\ dx \\
&= \int \frac{1}{2}dx + \int \frac{\cos 2x}{2}\ dx \\
&= \frac{x}{2} + c_1 + \frac{1}{2}\cdot\frac{1}{2}\sin 2x + c_2 \\
&= \frac{x}{2} + \frac{\sin 2x}{4} + c
\end{aligned}$$

Reducing degree of cosine and sine:

$\cos^2 x = \frac{1+\cos 2x}{2}$

$\sin^2 x = \frac{1-\cos 2x}{2}$

EXAMPLE 35 $\int \cot^2 x \, dx = ?$

Solution

$$\int \cot^2 x \, dx = \int (\cot^2 x + 1 - 1) \, dx$$

$$= \int (\cot^2 x + 1) \, dx - \int dx$$

$$= -\cot x + c_1 - x + c_2$$

$$= -\cot x - x - c$$

EXAMPLE 36 $\int \cos^2 3x \, dx - \int \sin^2 3x \, dx = ?$

Solution

$$\int \cos^2 3x \, dx - \int \sin^2 3x \, dx = \int (\cos^2 3x - \sin^2 3x) \, dx$$

$$= \int \cos 6x \, dx$$

$$= \frac{1}{6} \sin 6x + c$$

Double Angle Formulas:

$\cos 2x = \cos^2 x - \sin^2 x$

$\sin 2x = 2 \sin x \cos x$

EXAMPLE 37 $\int 3\tan^2(2x+1)\ dx = ?$

Solution $\int 3\tan^2(2x+1)\ dx = 3\int \tan^2(2x+1)\ dx$

$$= 3\int (1+\tan^2(2x+1)-1)\ dx$$

$$= 3(\frac{1}{2}\tan(2x+1) - x) + c$$

$$= \frac{3}{2}\tan(2x+1) - 3x + c$$

Check Yourself 5

Evaluate the integrals.

a. $\int (\sin x - \cos x)^2\ dx$

b. $\int (2\sin x - 3\cos x)$

c. $\int \frac{3}{\sqrt{4-4x^2}}\ dx$

d. $\int \left(\frac{1}{\cos^2 x} + \frac{1}{\sin^2 x}\right) dx$

e. $\int \sin 4x\ dx$

f. $\int \cos(5x-1)\ dx$

g. $\int \tan^2 2x$

h. $\int \cot^2(3x+1)\ dx$

i. $\int \sin^2 x\ dx$

j. $\int \frac{5}{1+x^2}\ dx$

Answers

a. $x + \frac{1}{2}\cos 2x + c$

b. $-2\cos x - 3\sin x + c$

c. $\frac{3}{2}\arcsin x + c$

d. $\tan x - \cot x + c$

e. $-\frac{1}{4}\cos 4x + c$

f. $\frac{1}{5}\sin(5x-1) + c$

g. $\frac{\tan 2x}{2} - x + c$

h. $-\frac{\cot(3x+1)}{3} - x + c$

i. $\frac{x}{2} - \frac{\sin x \cos x}{2} + c$

j. $5\arctan x + c$

Sir Isaac Newton (1642-1727)

Sir Isaac Newton is one of the greatest mathematicians in the history of mathematics.

Isaac was born in England in 1642. His father was a farmer but he died before Isaac was born. Isaac's mother remarried when Isaac was two years old and left Isaac in the care of his grandmother. Isaac went to Trinity College, Cambridge in 1661.

At Cambridge, Isaac want to be a lawyer, but first studied philosophy. However, when he learned about the work of Descartes in algebra and analytic geometry he decided to study mathematics and mechanics. He studied mathematics and physics until the University closed in 1665 because of the plague. Then he went home to Lincolnshire for two years.

While Isaac was at home he studied mathematics, optics, physics and astronomy. In this period of two years he established the beginnings of differential and integral calculus, independently of from the discoveries of another mathematician, Leibniz. He also made discoveries concerning the area of the region under a curve, the tangent of a curve at a point, and the maxima, minima and arc lengths of curves.

In 1669, at the age of twenty-seven, Isaac returned to Cambridge University and became a professor of mathematics. Then he studied optics. In 1672 he was elected to the Royal Society.

Newton's greatest works were about physics, mechanics and optics. He also discovered the rules of gravity, and made important discoveries in calculus.

Newton wrote his most famous book, the Principia Mathematica, *in 1687. In the* Principia, *Newton described his work in physics and its applications in astronomy and mathematics. Some people today consider the* Principia *to be one of the greatest science books ever written.*

In 1693, Newton suffered a nervous breakdown so he stopped his research and took up a government position in London. In 1703 he was elected President of the Royal Society. In 1708, Queen Anne knighted Newton, and he became Sir Isaac Newton. He was the first scientist to be so honored.

Isaac Newton died on 31 March 1727 in London.

EXERCISES 1.1

A. Definition of the Indefinite Integral

1. Evaluate the integrals.

a. $\int dw$

b. $\int dz$

c. $\int d\cos x$

d. $\int d(x^3 + 3x^2 - 1)$

B. Properties of the Indefinite Integral

2. Given $f(x) = \int d(x^2 - 1)$ and $f(1) = 2$, find $f(5)$.

3. $\int x \cdot f(x)\, dx = x^2 + 5x + 1$ is given. Find $f(x)$.

4. $\int x^3 \cdot f(x)\, dx = x^5 - 4x^3 - x + 1$ is given. Find $f(2)$.

5. $f'(x) = 5x^2 - 4x + 1$ and $f(1) = 3$ are given. Find $f(3)$.

6. Evaluate the integrals.

a. $\int (\cos x + 4x^3 - 2e^{2x})\, dx$

b. $\int (5x^3 - 3x^2 + 5)\, dx$

c. $\int \frac{5dx}{1+x^2}$

d. $\int 7x^8\, dx$

e. $\int \cos 4x\, dx$

f. $\int 7\sin x\, dx$

g. $\int 5e^{3x}\, dx$

h. $\int (2\cos 3x + 4\sin x - 4e^{5x})\, dx$

C. Basic Integration Formulas

7. Evaluate the integrals, using the basic formulas for integration.

a. $\int x^5\, dx$

b. $\int 4\, dx$

c. $\int x^{-3}\, dx$

d. $\int \frac{1}{x^5}\, dx$

e. $\int 3x^7\, dx$

f. $\int (\frac{1}{x^3} + \frac{1}{x^6} - \frac{1}{x^9})\, dx$

g. $\int 8e^{2x}\, dx$

h. $\int (3x^2 + 4x - 1)\, dx$

8. Evaluate the integrals, using the basic formulas for integration.

a. $\int \frac{\sqrt{x^3}}{x^3}\, dx$

b. $\int \frac{3x^2}{x^3}\, dx$

c. $\int (\frac{1}{x^2} + \frac{1}{x} + 1 + x)\, dx$

d. $\int (\sin x + \cos x)\ dx$

e. $\int \frac{x}{x^2}\ dx$

f. $\int \frac{5}{x+1}\ dx$

g. $\int \frac{1}{x-1}\ dx$

h. $\int \frac{4x^3+3x^2-4x+1}{x^2}\ dx$

9. Evaluate the integrals, using the basic formulas for integration.

a. $\int e^{2x}\ dx$

b. $\int e^{5x}\ dx$

c. $\int 3e^{2x}\ dx$

d. $\int 5e^{7x+2}\ dx$

e. $\int 7e^{x-2}\ dx$

f. $\int 2^{2x+1}\ dx$

g. $\int 5^x\ dx$

h. $\int 6^{2x-1}\ dx$

i. $\int 4^{3x-4}\ dx$

j. $\int 3^{3x}\ dx$

k. $\int 10^{x+1}\ dx$

l. $\int 4\cdot 3^{2x-1}\ dx$

10. Evaluate the integrals, using the basic formulas for integration.

a. $\int \sin 4x\ dx$

b. $\int \cos 5x\ dx$

c. $\int \frac{4}{\cos^2 x}\ dx$

d. $\int \frac{5}{\sin^2 2x}\ dx$

e. $\int 4\sec^2 4x\ dx$

f. $\int \tan^2 x\ dx$

g. $\int (\cot^2 x + 2)\ dx$

h. $\int \frac{3}{\sqrt{1-x^2}}\ dx$

i. $\int \frac{4}{x^2+1}\ dx$

j. $\int \frac{x^2+5}{x^2+1}\ dx$

k. $\int 5\cos(8x-4)\ dx$

l. $\int \frac{1}{\sqrt{1-4x^2}}\ dx$

m. $\int \frac{5}{9x^2+1}\ dx$

n. $\int \sin^2 x\ dx$

o. $\int \cot^2 x\ dx$

p. $\int (\tan^2 x - 1)\ dx$

2 INTEGRATION METHODS

We have seen how to use basic integral formulas and properties to find the integral of different functions. However, for some questions, using just these rules will not be enough to find the integral.

In this section, we will look at other methods we can use to integrate a function. These are:

integration by substitution,

integration by parts,

and special methods for the integration of rational, radical and trigonometric functions.

A. INTEGRATION BY SUBSTITUTION

For some integral problems, using x as variable does not give an expression that we can integrate easily. In this situation we can choose to change the variable. This method is called the substitution method of integration. The following theorem states the formula we use in the substitution method.

Theorem

Let $F(u)$ and $u(x)$ be two functions which are differentiable with respect to u and x respectively. Then

$$\int f(u(x)) \cdot u'(x)\, dx = F(u(x)) + c.$$

Proof We know from the Chain Rule that when $F'(x) = f(x)$ then $\dfrac{dF(u(x))}{dx} = F'(u(x)) \cdot u'(x)$.

Integrating both sides of this equation with respect to x gives us:

$$\int \frac{dF(u(x))}{dx}\, dx = \int f(u(x)) \cdot u'(x)\, dx.$$

This implies $F(u(x)) + c = \int f(u(x)) \cdot u'(x)\, dx$, which completes the proof.

In practical terms, we can summarize the substitution method of integration as follows:

SUBSTITUTION METHOD

1. Decide which term to substitute (i.e. select $u = g(x)$).
2. Differentiate both sides of $u = g(x)$ to get $\dfrac{du}{dx} = g'(x)$.
3. Rewrite the result as $du = g'(x)\, dx$.
4. Make these substitutions in the original integral to get a simpler expression.
5. Integrate the simpler expression, then substitute back the original terms using $u = g(x)$.

EXAMPLE 38 $\int (x-1)^2 \ dx = ?$

Solution 1 By using basic integration formula 1a we can get the answer:

$$\int (x-1)^2 \ dx = \int (x^2-2x+1) \ dx = \frac{x^3}{3} - x^2 + x + c_1.$$

Solution 2 We can use the substitution method:

Let $u = x - 1$, then $du = dx$. Then $\int (x-1)^2 \ dx = \int u^2 \ du = \frac{u^3}{3} + c_2$.

Now substitute back $u = x - 1$:

$$\frac{(x-1)^3}{3} + c_2 = \frac{x^3 - 3x^2 + 3x - 1}{3} + c_2 = \frac{x^3}{3} - x^2 + x - \frac{1}{3} + c_2.$$

We now have $\frac{1}{3} + c_2$ instead of c_1 but we can say the answers are the same. Can you see why?

Note

Any substitution will not work correctly in this method. An ideal substitution is the one that

1. removes all old variables,
2. makes the integral expression simpler.

EXAMPLE 39 $\int (1-x)^9 \ dx = ?$

Solution Let $u = 1 - x$, then $du = -dx$. Substitute this in the question:

$$\int (1-x)^9 \ dx = \int u^9 \ (-du) = -\int u^9 \ du = -\frac{u^{10}}{10} + c = -\frac{(1-x)^{10}}{10} + c.$$

EXAMPLE 40 $\int 3x^2 \cos x^3 dx = ?$

Solution Let $u = x^3$, then $du = 3x^2 \ dx$. Substituting these gives:

$$\int 3x^2 \cos x^3 \ dx = \int \cos u \ du = \sin u + c = \sin x^3 + c.$$

EXAMPLE 41 $\int (x^2+5)^7 \cdot x\,dx = ?$

Solution Let $u = x^2 + 5$, then $du = 2x\,dx$, i.e. $\dfrac{du}{2} = x\,dx$.

Substituting these gives:

$$\int (x^2+5)^7 \cdot x\,dx = \frac{1}{2}\int u^7 du = \frac{1}{2}\cdot\frac{u^8}{8} + c = \frac{(x^2+5)^8}{16} + c.$$

EXAMPLE 42 $\int (2x^2+1)(2x^3+3x)^7\,dx = ?$

Solution Let $u = 2x^3 + 3x$, then $du = (6x^2 + 3\,)\,dx$, i.e. $\dfrac{du}{3} = (2x^2+1)\,dx$. So

$$\int (2x^2+1)(2x^3+3x)^7\,dx = \int u^7\,\frac{du}{3} = \frac{1}{3}\int u^7\,du = \frac{1}{3}\frac{u^8}{8} + c = \frac{(2x^3+3x)^8}{24} + c.$$

EXAMPLE 43 $\int \dfrac{\ln x}{x}\,dx = ?$

Solution Let $u = \ln x$, then $du = \dfrac{1}{x}\cdot dx$. So

$$\int \frac{\ln x}{x}\,dx = \int u\,du = \frac{u^2}{2} + c = \frac{\ln^2 x}{2} + c.$$

EXAMPLE 44 $\int f^5(x)\cdot f'(x)\,dx = ?$

Solution Let $u = f(x)$, then $du = f'(x)\,dx$. So

$$\int f^5(x)\cdot f'(x)\,dx = \int u^5\,du = \frac{u^6}{6} + c = \frac{f^6(x)}{6} + c.$$

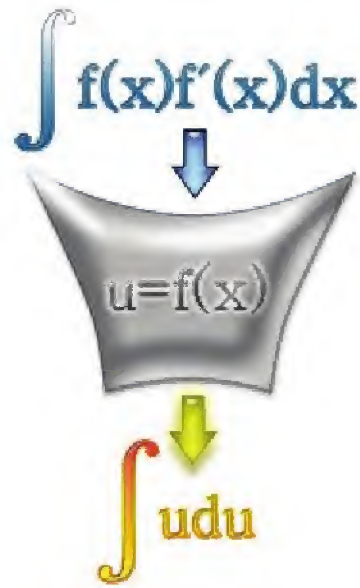
EXAMPLE 45 $\int \sqrt{1-3x}\, dx = ?$

Solution Let $u = 1 - 3x$, then $du = -3\,dx$, i.e. $dx = -\dfrac{du}{3}$.

So $\int \sqrt{1-3x}\, dx = \int \sqrt{u} \cdot (-\dfrac{du}{3})$

$$= -\frac{1}{3}\int u^{1/2}\, du = -\frac{1}{3}\cdot\frac{u^{3/2}}{\frac{3}{2}} + c$$

$$= -\frac{2}{9}\cdot(1-3x)^{3/2} + c.$$



EXAMPLE 46 $\int \sin^3 x \cdot \cos x\, dx = ?$

Solution Let $u = \sin x$, then $du = \cos x\, dx$. So

$$\int \sin^3 x \cdot \cos x\, dx = \int u^3\, du = \frac{u^4}{4} + c = \frac{\sin^4 x}{4} + c.$$

EXAMPLE 47 $\int \cos^3 x\, dx = ?$

Solution Let $u = \sin x$, then $du = \cos x\, dx$. So

$$\int \cos^3 x\, dx = \int \cos^2 x \cdot \cos x\, dx$$

$$= \int (1 - \sin^2 x)\cos x\, dx$$

$$= \int (1 - u^2)\, du$$

$$= u - \frac{u^3}{3} + c$$

$$= \sin x - \frac{\sin^3 x}{3} + c.$$

Fundamental Trigonometric Identity: $\sin^2 x + \cos^2 x = 1$

EXAMPLE 48 $\int \frac{e^x}{\sqrt{1-e^{2x}}}\,dx = ?$

Solution Let $u = e^x$ and $du = e^x\,dx$. So

$$\int \frac{e^x}{\sqrt{1-e^2x}}\,dx = \int \frac{du}{\sqrt{1-u^2}}$$

$$= \arcsin u + c_1 = \arcsin e^x + c_1$$

or

$$= -\arccos u + c_2 = -\arccos e^x + c_2.$$

EXAMPLE 49 $\int 3^{3x^3+4x^2-1}\cdot(9x^2+8x)\,dx = ?$

Solution Let $u = 3x^3 + 4x^2 - 1$, then $du = (9x^2 + 8x)\,dx$.

$$\int 3^{3x^3+4x^2-1}\cdot(9x^2+8x)\,dx = \int 3^u\,du$$

$$= \frac{3^u}{\ln 3} + c$$

$$= \frac{3^{3x^3+4x^2-1}}{\ln 3} + c.$$

EXAMPLE 50 $\int \frac{1}{\sqrt{1-9x^2}}\,dx = ?$

Solution Let $u = 3x$, then $du = 3\,dx$, i.e. $dx = \frac{du}{3}$. Substituting gives:

$$\int \frac{1}{\sqrt{1-9x^2}}\,dx = \int \frac{1}{\sqrt{1-u^2}}\frac{du}{3}$$

$$= \frac{1}{3}\arcsin u + c_1 = \frac{1}{3}\arcsin 3x + c_1$$

or

$$= -\frac{1}{3}\arccos u + c_2 = -\frac{1}{3}\arccos 3x + c_2.$$

EXAMPLE 51 $\int \frac{1}{1+4x^2}\,dx = ?$

Solution Let $u = 2x$, then $du = 2\,dx$, i.e. $dx = \frac{du}{2}$. So

$$\int \frac{1}{1+4x^2}\,dx = \int \frac{1}{1+u^2}\frac{du}{2}$$

$$= \frac{1}{2}\arctan u + c_1 = \frac{1}{2}\arctan 2x + c_1$$

or

$$= -\frac{1}{2}\operatorname{arccot} u + c_2 = -\frac{1}{2}\operatorname{arccot} 2x + c_2.$$

EXAMPLE 52 $\int \frac{\sin(\ln x)}{x}\,dx = ?$

Solution Let $u = \ln x$, then $du = \frac{1}{x}\,dx$. So

$$\int \frac{\sin(\ln x)}{x}\,dx = \int \sin u\,du$$

$$= -\cos u + c$$

$$= -\cos(\ln x) + c.$$

EXAMPLE 53 $\int (x+2)(x-1)^4\,dx = ?$

Solution In this problem, we cannot immediately get the answer using one substitution. Let us find a substitution for each term instead.

Let $u = x - 1$ so $du = dx$.

Now we can write $x = u+1$, so $x+2 = u + 3$. Now we can substitute:

$$\int (x+2)(x-1)^4\,dx = \int (u+3)\cdot u^4\,du$$

$$= \int (u^5 + 3u^4)\,du$$

$$= \frac{u^6}{6} + 3\cdot\frac{u^5}{5} + c$$

$$= \frac{(x-1)^6}{6} + \frac{3(x-1)^5}{5} + c.$$

EXAMPLE 54 $\int \frac{7}{(5x+2)^9}\,dx = ?$

Solution Let $u = 5x+2$, then $du = 5\,dx$, i.e. $dx = \frac{du}{5}$.

$$\text{So } \int \frac{7}{(5x+2)^9}\,dx = \int \frac{7}{5u^9}\,du$$
$$= \frac{7}{5}\int u^{-9}\,du$$
$$= \frac{7}{5}\cdot\frac{u^{-8}}{-8} + c$$
$$= -\frac{7}{40u^8} + c$$
$$= -\frac{7}{40(5x+2)^8} + c.$$

Check Yourself 6

Evaluate the integrals.

a. $\int \sin(1-x)\,dx$ b. $\int (1-x^3)^5 \cdot x^2\,dx$ c. $\int x\cdot\sin(5x^2-1)\,dx$ d. $\int \frac{5}{1+9x^2}\,dx$

e. $\int e^{\cos x}\sin x\,dx$ f. $\int \frac{\sin x}{1+\cos^2 x}\,dx$ g. $\int \sin(7x+1)\,dx$ h. $\int 5^{x^2+4x-2}\cdot(x+2)\,dx$

i. $\int \frac{1}{1+2x^2}\,dx$ j. $\int \frac{3}{\sqrt{1-4x^2}}\,dx$ k. $\int \frac{x}{1+x^4}\,dx$ l. $\int \sin x\cos^3 x\,dx$

Answers

a. $\cos(x-1) + c$ b. $-\frac{(x^3-1)^6}{18} + c$ c. $-\frac{\cos(5x^2-1)}{10} + c$ d. $\frac{5\arctan 3x}{3} + c$

e. $-e^{\cos x} + c$ f. $-\arctan(\cos x) + c$ g. $-\frac{1}{7}\cos(7x+1) + c$ h. $\frac{5^{x^2+4x-2}}{2\ln 5} + c$

i. $\frac{\sqrt{2}\arctan(x\sqrt{2})}{2} + c$ j. $\frac{3\arcsin 2x}{2} + c$ k. $\frac{\arctan x^2}{2} + c$ l. $-\frac{\cos^4 x}{4} + c$

B. INTEGRATION BY PARTS

Integration by parts is the second main method of integration. Many of the other methods of integration use integration by parts, so it is an important method to master.

Integration by parts is a method for evaluating integrals of the form $\int f(x)\cdot g(x)\,dx$ which are difficult to evaluate using other techniques. The method uses the following theorem:

Theorem

Let $u = f(x)$ and $v = g(x)$ be two differentiable functions with respect to x, then

$$\int u\cdot v'\,dx = u\cdot v - \int v\cdot u'\,dx.$$

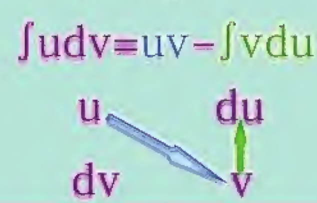
Proof We know from differentiation that $d(u\cdot v) = v\cdot du + u\cdot dv$.

If we take the integral of both sides we get $\int d(u\cdot v) = \int v\,du + \int u\,dv$, which gives $u\cdot v = \int v\,du + \int u\,dv$.

Rearranging gives $\int u\,dv = u\cdot v - \int v\,du$, which is the required result.

SUBSTITUTION METHOD

1. Express the integrand as a product of two expressions such that
   a. one of them is easy to differentiate (u)
   b. the other is easy to integrate (dv).
2. Use the below scheme to rewrite the integral.



Note

In this part we use u' and v' instead of du and dv to make expressions shorter.

EXAMPLE 55 $\int x\cdot e^x\,dx = ?$

Solution Let $u = x$ and $v' = e^x$, then $u' = 1$ and $v = e^x$.

So $\int x\cdot e^x\,dx = x\cdot e^x - \int e^x\,dx = x\cdot e^x - e^x + c$.

Here note that although $v' = e^x$ we did not write $v = e^x + c_1$ but just $v = e^x$. This is because of the fact that after evaluating the final integral there will be a constant which is the sum of all constants.

Note

In this method there is no rule for the selection of u and v' but generally we choose u to be the function whose degree reduces when we take its derivative. Typically, we often choose logarithmic and inverse trigonometric functions for u, and functions such as e^x, $\sin x$, $\cos x$ etc. for v', although these are only guidelines.

EXAMPLE 56 $\int x^2 \cdot \ln x \, dx = ?$

Solution Let $u = \ln x$ and $v' = x^2$, then $u' = \dfrac{1}{x}$ and $v = \dfrac{x^3}{3}$.

So $\int x^2 \cdot \ln x \, dx = \ln x \cdot \dfrac{x^3}{3} - \int \dfrac{x^3}{3} \cdot \dfrac{1}{x} dx = \ln x \cdot \dfrac{x^3}{3} - \dfrac{1}{3} \cdot \dfrac{x^3}{3} + c = \dfrac{x^3}{3}(\ln x - \dfrac{1}{3}) + c.$

EXAMPLE 57 $\int \dfrac{\ln x}{x^4} \, dx = ?$

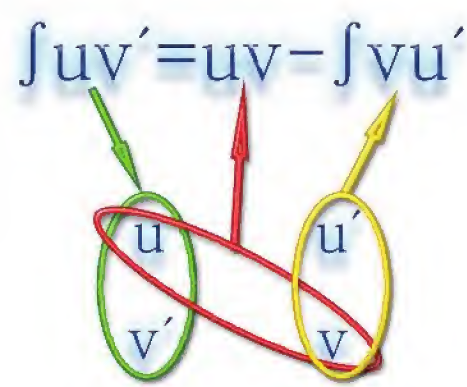
Solution Let $u = \ln x$ and $v' = \dfrac{1}{x^4}$, then $u' = \dfrac{1}{x}$ and $v = -\dfrac{1}{3x^3}$.

So $\int \dfrac{\ln x}{x^4} \, dx = -\dfrac{1}{3x^3} \cdot \ln x - \int -\dfrac{1}{3x^3} \cdot \dfrac{1}{x} \, dx$

$$= -\frac{\ln x}{3x^3} + \frac{1}{3}\int x^{-4} \, dx$$

$$= -\frac{\ln x}{3x^3} + \frac{1}{3}(-\frac{1}{3x^3}) + c$$

$$= -\frac{\ln x}{3x^3} - \frac{1}{9x^3} + c.$$



EXAMPLE 58 $\int x \cdot \cos x \, dx = ?$

Solution Let $u = x$ and $v' = \cos x$, then $u' = 1$ and $v = \sin x$.

So $\int x \cdot \cos x \, dx = x \cdot \sin x - \int \sin x \, dx = x \cdot \sin x + \cos x + c.$

EXAMPLE 59 $\int \arctan x\, dx = ?$

Solution Let $u = \arctan x$ and $v' = 1$, then $u' = \dfrac{1}{1+x^2}$ and $v = x$.

So $\int \arctan x\, dx = x \cdot \arctan x - \int \dfrac{x}{1+x^2}\, dx$. We can now use integration by substitution:

Let $w = 1+x^2$, then $dw = 2x\, dx$ and $x \cdot dx = \dfrac{dw}{2}$. This gives

$$\int \arctan x\, dx = x \cdot \arctan x - \int \frac{1}{2w}\, dw$$

$$= x \cdot \arctan x - \frac{1}{2}\ln|w| + c$$

$$= x \cdot \arctan x - \frac{1}{2}\ln(1+x^2) + c$$

EXAMPLE 60 $\int x \cdot \sin 4x\, dx = ?$

Solution Let $u = x$ and $v' = \sin 4x$, then $u' = 1$ and $v = -\dfrac{1}{4}\cos 4x$.

$$\int x \cdot \sin 4x\, dx = -\frac{x \cdot \cos 4x}{4} - \int -\frac{\cos 4x}{4}\, dx$$

$$= -\frac{x \cdot \cos 4x}{4} + \frac{1}{4}\int \cos 4x\, dx$$

$$= -\frac{x \cdot \cos 4x}{4} + \frac{\sin 4x}{16} + c$$

EXAMPLE 61 $\int x\sqrt{x-2}\, dx = ?$

Solution Let $u = x$ and $v' = \sqrt{x-2}$, then $u' = 1$ and $v = \dfrac{2}{3}(x-2)^{3/2}$. This gives

$$\int x\sqrt{x-2}\, dx = \frac{2}{3}(x-2)^{3/2}x - \int \frac{2}{3}(x-2)^{3/2}\, dx$$

$$= \frac{2}{3}(x-2)^{3/2}x - \frac{2}{3} \cdot \frac{(x-2)^{5/2}}{\frac{5}{2}} + c$$

$$= \frac{2}{3}(x-2)^{3/2}x - \frac{4(x-2)^{5/2}}{15} + c.$$

Note

When we integrate a function, we may get two different answers if we use two different methods. However, if our working is correct for each method then we say that both of the solutions are correct. This is because the integral is the antiderivative, and there may be two or more functions with the same derivative.

EXAMPLE 62 $\int e^{2x} \cdot \sin e^x \; dx = ?$

Solution Let $u = e^x$ and $v' = e^x \sin e^x$, then $u' = e^x$. To find v we need to integrate $e^x \sin e^x$:

Let $t = e^x$, then $dt = e^x\,dx$ and so $v = \int e^x \sin e^x \; dx = \int \sin t \; dt = -\cos t = -\cos e^x$.

$$\int e^{2x} \cdot \sin e^x \; dx = -e^x \cos e^x - \int -e^x \cos e^x \; dx = -e^x \cos e^x + \int e^x \cos e^x \; dx \quad (1)$$

We now need to integrate $e^x \cos e^x$. Let us use $k = e^x$ and $dk = e^x\,dx$.

By substitution, $\int e^x \cos e^x \; dx = \int \cos k \; dk = \sin k + c = \sin e^x + c. \quad (2)$

Combining (1) and (2) gives $\int e^{2x} \cdot \sin e^x \; dx = -e^x \cos e^x + \sin e^x + c.$

EXAMPLE 63 $\int \sin(\ln x) \; dx = ?$

Solution Let $u = \sin(\ln x)$ and $v' = 1$, then $u' = \dfrac{\cos(\ln x)\,dx}{x}$ and $v = x$.

This gives $\int \sin(\ln x) \; dx = x \cdot \sin(\ln x) - \int x \cdot \dfrac{\cos(\ln x)}{x} \; dx = x \cdot \sin(\ln x) - \int \cos(\ln x) \; dx. \quad (1)$

Now let us evaluate $\int \cos(\ln x)\;dx$ separately:

Choosing $u_1 = \cos(\ln x)$ and $dv_1' = 1$ gives $du_1 = -\dfrac{\sin(\ln x)\,dx}{x}$ and $v_1 = x$, so

$$\int \cos(\ln x) \; dx = x \cdot \cos(\ln x) - \int x \cdot \left(-\frac{\sin(\ln x)}{x}\right) dx$$

$$= x \cdot \cos(\ln x) + \int \sin(\ln x) \; dx. \quad (2)$$

Substituting (2) in (1) gives

$$\int \sin(\ln x) \; dx = x \cdot \sin(\ln x) - \left(x \cdot \cos(\ln x) + \int \sin(\ln x) \; dx\right)$$

$$= x \cdot \sin(\ln x) - x\cos(\ln x) - \int \sin(\ln x) \; dx$$

$$2 \cdot \int \sin(\ln x) \; dx = x \cdot \sin(\ln x) - x \cdot \cos(\ln x),$$

$$\int \sin(\ln x) \; dx = \frac{x(\sin(\ln x) - \cos(\ln x))}{2} + c.$$

EXAMPLE 64 $\int e^x \sin x \, dx = ?$

Solution Let $u = e^x$ and $v' = \sin x$, then $u' = e^x$ and $v = -\cos x$. This gives

$\int e^x \sin x \, dx = -e^x \cos x + \int e^x \cos x \, dx.$

For the second part, we will use integration by parts once more.

Let $u_1 = e^x$ and $v_1' = \cos x$, then $u_1' = e^x$ and $v_1 = \sin x$. So

$\int e^x \cos x \, dx = e^x \sin x - \int e^x \sin x \, dx$ and

$$\int e^x \sin x \, dx = -e^x \cos x + \int e^x \cos x \, dx$$

$$= -e^x \cos x + e^x \sin x - \int e^x \sin x \, dx$$

$$2\int e^x \sin x \, dx = -e^x \cos x + e^x \sin x$$

$$\int e^x \sin x \, dx = \frac{e^x}{2}(\sin x - \cos x) + c.$$

Check Yourself 7

Evaluate the integrals.

a. $\int \ln x \, dx$
b. $\int x^2 \cdot \sin x \, dx$
c. $\int x \cdot \ln x \, dx$
d. $\int x^2 \cdot e^{2x} \, dx$
e. $\int \arcsin x \, dx$
f. $\int \log(x+3) \, dx$
g. $\int (2x+1)\sin 3x \, dx$
h. $\int e^{3x} \cdot \sin x \, dx$
i. $\int e^{(2x+1)} \cdot \cos(x-1) \, dx$
j. $\int (x-1) \cdot \ln x \, dx$
k. $\int x\sqrt{x+1} \, dx$
l. $\int 3x \cdot \sin x \cdot \cos x \, dx$

Answers

a. $x \ln x - x + c$
b. $(2 - x^2) \cos x + 2x \sin x + c$
c. $\frac{x^2 \ln x}{2} - \frac{x^2}{4} + c$
d. $\frac{e^{2x}(2x^2 - 2x + 1)}{4} + c$
e. $x \arcsin x + \sqrt{1 - x^2} + c$
f. $(x + 3) \cdot \log(x + 3) - \frac{x}{\ln 10} + c$
g. $\frac{2\sin 3x}{9} - \frac{(2x+1)\cos 3x}{3} + c$
h. $e^{3x}(\frac{3\sin x}{10} - \frac{\cos x}{10}) + c$
i. $e^{2x+1}(\frac{2\cos(x-1)}{5} + \frac{\sin(x-1)}{5}) + c$
j. $(\frac{x^2}{2} - x)\ln x - \frac{x(x-4)}{4} + c$
k. $\frac{2(x+1)^{3/2}(3x-2)}{15} + c$
l. $\frac{3\sin x \cos x}{4} + \frac{3x\sin^2 x}{2} - \frac{3x}{4} + c$

C. INTEGRATING PARTIAL FRACTIONS

We use different methods to evaluate integrals of the form $\int \frac{P(x)}{Q(x)} dx$, where $P(x)$ and $Q(x)$ are polynomials and $Q(x) \neq 0$.

The method we choose depends on the partial fraction involved. Let us look at the main possibilities.

1. $\int \frac{P(x)}{Q(x)}$ with $\deg[P(x)] = \deg[Q(x)] - 1$

Given a polynomial $P(x)$ to denote its degree we use $\deg[P(x)]$.

For integrals of this type we use the substitution $u = Q(x)$ and try to find du in terms of $P(x)\ dx$. After this, we try to find the answer.

EXAMPLE 65 $\int \frac{3}{5x+1}\ dx = ?$

Solution Let $u = 5x+1$, then $du = 5\ dx$, i.e. $dx = \frac{du}{5}$.

So $\int \frac{3}{5x+1} dx = \int \frac{3}{5u}\ du = \frac{3}{5} \cdot \ln|u| + c = \frac{3}{5} \cdot \ln|5x+1| + c.$

EXAMPLE 66 $\int \frac{2x-3}{x^2-3x-1} dx = ?$

Solution Let $u = x^2 - 3x - 1$, then $du = (2x-3)\ dx$.

So $\int \frac{2x-3}{x^2-3x-1} dx = \int \frac{1}{u} du = \ln|u| + c = \ln|x^2-3x-1| + c.$

EXAMPLE 67 $\int \frac{1}{x^2+2x+1} dx = ?$

Solution $\int \frac{1}{x^2+2x+1} dx = \int \frac{1}{(x+1)^2} dx$

Let $u = x + 1$, then $du = dx$.

$$\int \frac{1}{(x+1)^2} dx = \int \frac{1}{u^2} du = \int u^{-2}\ du = -\frac{1}{u} + c = -\frac{1}{x+1} + c$$

Check Yourself 8

Evaluate the integrals.

a. $\int \frac{4}{7x-6}\,dx$ b. $\int \frac{2x+3}{x^2+3x-1}\,dx$ c. $\int \frac{3}{(x+1)^4}dx$

d. $\int \frac{dx}{x^2+8x+16}$ e. $\int \frac{x^2-2x}{x^3-3x^2-1}\,dx$

Answers

a. $\frac{4\ln|7x-6|}{7}+c$ b. $\ln|x^2+3x-1|+c$ c. $-\frac{1}{(x+1)^3}+c$

d. $-\frac{1}{x+4}+c$ e. $\frac{\ln|x^3-3x^2-1|}{3}+c$

A polynomial $Q(x)$ is reducible in R if it is possible to factorize it.

2. $\int \frac{P(x)}{Q(x)}$ with $\deg[P(x)] < \deg[Q(x)]$ and $Q(x)$ reducible in R

In this case, if $Q(x)$ is linear (degree 1) then we can evaluate the integral easily using the formula $\int \frac{u'}{u}\,dx = \ln|u| + c$.

However, if $\deg[Q(x)] > 1$ we begin by trying to write given expression as the sum or difference of two or more partial fractions. The rules for doing this are given below.

Rule

1. $$\frac{P(x)}{(ax+b)\cdot(cx+d)} = \frac{A}{ax+b}+\frac{B}{cx+d}$$

2. $$\frac{P(x)}{(ax+b)^n} = \frac{A}{ax+b}+\frac{B}{(ax+b)^2}+\frac{C}{(ax+b)^3}+\ldots+\frac{D}{(ax+b)^n}$$

3. $$\frac{P(x)}{(ax+b)\cdot(cx^2+dx+e)} = \frac{A}{ax+b}+\frac{Bx+C}{cx^2+dx+e}$$

Notice that the number of partial fractions in each expansion is the same as the number of factors in the denominator of the original fraction.

EXAMPLE 68 $\int \frac{5x+7}{(x-2)(3x+1)}dx = ?$

Solution We begin by writing the integrand as the sum of partial fractions:

$$\frac{5x+7}{(x-2)(3x+1)} = \frac{A}{x-2} + \frac{B}{3x+1} = \frac{3Ax+A+Bx-2B}{(x-2)(3x+1)} = \frac{(3A+B)x+(A-2B)}{(x-2)(3x+1)}.$$

Solving for A and B gives

$$3A + B = 5$$

$$A - 2B = 7, \text{ so}$$

$A = \frac{17}{7}$ and $B = -\frac{16}{7}$, and $\frac{5x+7}{(x-2)(3x+1)} = \frac{17}{7\cdot(x-2)} - \frac{16}{7\cdot(3x+1)}$. This is the sum.

Now integrate both sides: $\int \frac{5x+7}{(x-2)(3x+1)}dx = \int \frac{17}{7\cdot(x-2)}dx - \int \frac{16}{7\cdot(3x+1)}dx$

$$= \frac{17}{7}\cdot \ln|x-2| - \frac{16}{21}\cdot \ln|3x+1| + c.$$

Note

We can also find A and B using the following method:

$$\frac{5x+7}{(x-2)(3x+1)} = \frac{A}{x-2} + \frac{B}{3x+1}$$

$$\frac{5x+7}{3x+1} = A + \frac{(x-2)\cdot B}{3x+1} \quad \text{multiply both sides by } (x-2)$$

$$\frac{5\cdot 2+7}{3\cdot 2+1} = A + \frac{0\cdot B}{3x+1} \quad \text{replace } x = 2 \ \ (\text{to make } x-2=0)$$

which gives $A = \frac{17}{7}$.

We can use the same method to find $B = -\frac{16}{7}$, and then complete the integration as in Example 68.

EXAMPLE 69 $\int \frac{11x+4}{2x^2+x-3}dx = ?$

Solution First we factorize the denominator:

$2x^2 + x - 3 = (x-1)\cdot(2x+3).$

So $\frac{11x+4}{2x^2+x-3} = \frac{A}{x-1} + \frac{B}{2x+3} = \frac{2Ax+3A+Bx-B}{(x-1)(2x+3)} = \frac{(2A+B)x+(3A-B)}{2x^2+x-3}$, which gives

$$2A + B = 11$$
$$3A - B = 4.$$

The solution of this system gives $A = 3$ and $B = 5$, so we have $\frac{11x+4}{2x^2+x-3} = \frac{3}{x-1} + \frac{5}{2x+3}$.

Now integrate both sides: $\int \frac{11x+4}{2x^2+x-3}dx = \int \frac{3}{x-1}dx + \int \frac{5}{2x+3}dx$

$$= 3\cdot\ln|x-1| + \frac{5}{2}\ln|2x+3| + c.$$

EXAMPLE 70 $\int \frac{x+1}{x^3-1}dx = ?$

Solution Factorize the denominator: $x^3 - 1 = (x-1)\cdot(x^2+x+1)$.

This gives

$$\frac{x+1}{x^3-1} = \frac{x+1}{(x-1)(x^2+x+1)} = \frac{A}{x-1} + \frac{Bx+C}{x^2+x+1} = \frac{Ax^2+Ax+A+Bx^2-Bx+Cx-C}{x^3-1}$$

$$= \frac{(A+B)x^2+(A-B+C)x+(A-C)}{x^3-1}$$

$$A + B = 0$$
$$A - B + C = 1$$
$$A - C = 1.$$

Solving this system gives us $A = \frac{2}{3}$, $B = -\frac{2}{3}$, $C = -\frac{1}{3}$, so we have

$$\frac{x+1}{x^3-1} = \frac{2}{3(x-1)} - \frac{1}{3}\cdot\frac{2x+1}{x^2+x+1}.$$

Now integrate both sides: $\int \frac{x+1}{x^3-1} = \int \frac{2}{3(x-1)}dx + \int -\frac{1}{3}\cdot\frac{2x+1}{x^2+x+1}dx$

$$= \frac{2}{3}\cdot\ln|x-1| + c_1 + \int -\frac{1}{3}\cdot\frac{2x+1}{x^2+x+1}dx. \quad (1)$$

We can evaluate the remaining integral using the substitutions

$u = x^2 + x + 1$ and $du = (2x + 1)\,dx$:

$$\int -\frac{1}{3}\cdot\frac{2x+1}{x^2+x+1} = -\frac{1}{3}\int\frac{du}{u} = -\frac{1}{3}\cdot\ln|\,u\,| + c_2$$

$$= -\frac{1}{3}\cdot\ln|\,x^2+x+1\,| + c_2. \quad (2)$$

Combining (1) and (2) gives $\int\frac{x+1}{x^3-1}\,dx = \ln\left|\frac{x^2-2x+1}{x^2+x+1}\right|^{1/3} + c.$

EXAMPLE 71 $\int\frac{5x-1}{(2x+3)^2}\,dx = ?$

Solution $\frac{5x-1}{(2x+3)^2} = \frac{A}{2x+3} + \frac{B}{(2x+3)^2} = \frac{2Ax+3A+B}{(2x+3)^2}$

$$2A = 5$$
$$3A + B = -1$$

Solving this system gives $A = \frac{5}{2}$ and $B = -\frac{17}{2}$, so

$$\int\frac{5x-1}{(2x+3)^2}\,dx = \int\frac{5}{2\cdot(2x+3)}\,dx - \int\frac{17}{2(2x+3)^2}\,dx.$$

Use the substitutions $u = 2x + 3$ and $du = 2dx$ in the second part:

$$\int\frac{5x-1}{(2x+3)^2}dx = \frac{5}{4}\ln|\,2x+3\,| + c_1 - \frac{17}{2}\int\frac{1}{2}\cdot u^{-2}du = \frac{5}{4}\ln|\,2x+3\,| + \frac{17}{4(2x+3)} + c.$$

EXAMPLE 72 $\int\frac{e^x}{(e^x+2)(e^x-3)}\,dx = ?$

Solution In this problem, the denominator $Q(x)$ is not immediately reducible in R. First we need to change the given expression to a rational function, then by using partial fractions we will be able to evaluate the integral.

Let $u = e^x$ so $du = e^x\,dx$ to give a rational function:

$$\int\frac{e^x}{(e^x+2)(e^x-3)}\,dx = \int\frac{1}{(u+2)(u-3)}\,du.$$

Now use partial fractions:

$\frac{1}{(u+2)(u-3)} = \frac{A}{u+2} + \frac{B}{u-3} = \frac{(A+B)u-3A+2B}{(u+2)(u-3)}$. This gives

$$A + B = 0$$
$$-3A + 2B = 1.$$

Solving this system gives us $A = -\frac{1}{5}$ and $B = \frac{1}{5}$, i.e.

$$\int \frac{du}{(u+2)(u-3)} = \int \frac{-1}{5(u+2)}\,du + \int \frac{1}{5(u-3)}\,du = -\frac{1}{5}\ln|u+2| + \frac{1}{5}\ln|u-3| + c.$$

So we get the result:

$$\int \frac{e^x}{(e^x+2)(e^x-3)}\,dx = -\frac{1}{5}\ln|e^x+2| + \frac{1}{5}\ln|e^x-3| + c = \frac{1}{5}\ln\left|\frac{e^x-3}{e^x+2}\right| + c$$

3. $\int \frac{P(x)}{Q(x)}$ with deg[P(x)] < deg[Q(x)] and Q(x) not reducible in R

The expression $\int \frac{1}{ax^2+bx+c}\,dx$ is given. If $\Delta = b^2 - 4ac < 0$ then we can use the following method to evaluate the integral:

$$\int \frac{1}{ax^2+bx+c}dx = \int \frac{dx}{(mx+n)^2+r^2} = \frac{1}{r^2}\int \frac{dx}{\left(\frac{mx+n}{r}\right)^2+1}$$

$$= \frac{1}{r^2}\cdot\frac{r}{m}\cdot\arctan\left(\frac{mx+n}{r}\right) + c_1 = \frac{1}{rm}\arctan\left(\frac{mx+n}{r}\right) + c_1$$

$$\text{or} \quad = -\frac{1}{rm}\operatorname{arccot}\left(\frac{mx+n}{r}\right) + c_2.$$

EXAMPLE 73 Evaluate $\int \frac{1}{x^2+4x+5}\,dx$.

Solution $\int \frac{dx}{x^2+4x+5} = \int \frac{dx}{x^2+4x+4+1} = \int \frac{dx}{(x+2)^2+1} = \arctan(x+2) + c$

or $= -\operatorname{arccot}(x+2) + c$

4. $\int \frac{P(x)}{Q(x)}$ with deg[P(x)] ≥ deg[Q(x)]

In this situation, we first divide the numerator by the denominator, then calculate the integrals separately.

EXAMPLE 74 $\int \frac{4x^2+14x+3}{x^2+3x}\,dx = ?$

Solution We have deg $|P(x)|$ = deg $|Q(x)|$, so we divide the numerator by the denominator:

$$\frac{4x^2+14x+3}{x^2+3x} = 4+\frac{2x+3}{x^2+3x}.$$

Integrate both sides: $\int \frac{4x^2+14x+3}{x^2+3x}\,dx = \int 4\,dx + \int \frac{2x+3}{x^2+3x}\,dx = 4x + c_1 + \int \frac{2x+3}{x^2+3x}\,dx.$

We can evaluate the remaining integral using the substitutions $u = x^2 + 3x$ and $du = (2x + 3)\,dx$:

$$\int \frac{2x+3}{x^2+3x}\,dx = \int \frac{du}{u} = \ln|u| + c_2.$$

So $\int \frac{4x^2+14x+3}{x^2+3x}\,dx = 4x + c_1 + \ln|u| + c_2 = 4x + \ln|x^2+3x| + c.$

EXAMPLE 75 $\int \frac{x^3+3x}{x^2+1}\,dx = ?$

Solution We have deg $|P(x)|$ > deg $|Q(x)|$, so we divide the numerator by the denominator:

$$\frac{x^3+3x}{x^2+1} = x+\frac{2x}{x^2+1}.$$

Integrate both sides: $\int \frac{x^3+3x}{x^2+1}\,dx = \int x\,dx + \int \frac{2x}{x^2+1}\,dx = \frac{x^2}{2} + c_1 + \int \frac{2x}{x^2+1}\,dx.$

We can evaluate the remaining integral using the substitutions $u = x^2 + 1$ and $du = 2x\,dx$,

which give $\int \frac{x^3+3x}{x^2+1}\,dx = \frac{x^2}{2} + c_1 + \ln|u| + c_2 = \frac{x^2}{2} + \ln(x^2+1) + c.$

EXAMPLE 76 $\int \frac{3x-1}{x+2}\,dx = ?$

Solution 1 We have deg $|P(x)|$ = deg $|Q(x)|$, so we divide the numerator by the denominator:

$$\frac{3x-1}{x+2} = 3-\frac{7}{x+2}.$$

Integrate both sides: $\int \frac{3x-1}{x+2}\,dx = \int (3-\frac{7}{x+2})\,dx = 3x - 7\ln|x+2| + c.$

Solution 2 We can use normal substitution twice.

Let $u = x + 2$ and $du = dx$.

This gives $x = u - 2$, i.e. $3x - 1 = 3(u - 2) - 1 = 3u - 7$. So

$$\int \frac{3x-1}{x+2}\,dx = \int \frac{3u-7}{u}\,du = \int (3 - \frac{7}{u})\,du = 3u - 7\ln|u| + c = 3(x+2) - 7\ln|x+2| + c$$

$$= 3x - 7\cdot\ln|x+2| + c.$$

Check Yourself 9

Evaluate the integrals.

a. $\int \frac{1}{(x-1)(x+2)}\,dx$ b. $\int \frac{2x-1}{(x+1)(x+2)}\,dx$ c. $\int \frac{x-1}{(x+1)(x-2)^2}\,dx$

d. $\int \frac{x^2+3}{x^2+1}\,dx$ e. $\int \frac{3x-1}{x^2-1}\,dx$ f. $\int \frac{1}{x^2+2x+5}\,dx$

g. $\int \frac{x^2-1}{x^2-16}\,dx$ h. $\int \frac{1-x}{x^2+3x}\,dx$ i. $\int \frac{3}{e^x-2}\,dx$

Answers

a. $\frac{1}{3}\ln|\frac{x-1}{x+2}| + c$ b. $5\ln|x+2| - 3\ln|x+1| + c$ c. $\frac{2}{9}\ln|\frac{x-2}{x+1}| - \frac{1}{3(x-2)} + c$

d. $2\arctan x + x + c$ e. $\ln|x-1| + 2\cdot\ln|x+1| + c$ f. $\frac{1}{2}\arctan(\frac{x+1}{2}) + c$

g. $\frac{15}{8}\ln|\frac{x-4}{x+4}| + x + c$ h. $\frac{1}{3}\ln|x| - \frac{4}{3}\ln|x+3| + c$ i. $\frac{3}{2}\ln|e^x-2| - \frac{3x}{2} + c$

D. INTEGRATING RADICAL FUNCTIONS

The integration of functions of the form $\int \sqrt{f(x)}\,dx$, $\int \sqrt[n]{f(x)}\,dx$, $\int \sqrt{a^2 \pm u^2}$ requires the use of special methods. Let us look at these methods.

1. Integrating Simple Radical Functions

There are many different types of radical function, and we can use different methods to integrate them. In this section we will concentrate on radical functions that can be integrated easily using the methods we have studied. We call these functions simple radical functions.

When integrating a simple radical function, we first try to eliminate the radical sign. For this reason we use substitutions such as u^2, u^3, etc. depending on the degree of the root.

EXAMPLE 77 $\int \sqrt{3x+1}\, dx = ?$

Solution To eliminate the root we can substitute $u^2 = 3x + 1$ $(x \geq -\frac{1}{3})$, so $2u\, du = 3\, dx$ and $dx = \frac{2u\, du}{3}$.

$$\text{Then } \int \sqrt{3x+1}\, dx = \int \sqrt{u^2} \cdot \frac{2u\, du}{3} = \int \frac{u \cdot 2u\, du}{3} = \frac{2}{3}\int u^2\, du = \frac{2}{3} \cdot \frac{u^3}{3} + c = \frac{2(3x+1)^{\frac{3}{2}}}{9} + c.$$

EXAMPLE 78 $\int \sqrt[3]{5x-2}\, dx = ?$

Solution To eliminate the third degree root we choose $u^3 = 5x - 2$, then

$3u^2\, du = 5\, dx$, i.e. $dx = \frac{3u^2}{5}\, du$.

$$\text{So } \int \sqrt[3]{5x-2}\, dx = \int \sqrt[3]{u^3} \cdot \frac{3u^2}{5}\, du = \int \frac{3u^3}{5}\, du = \frac{3}{5} \cdot \frac{u^4}{4} + c = \frac{3(5x-2)^{\frac{4}{3}}}{20} + c.$$

EXAMPLE 79 $\int \frac{x}{\sqrt{x^2+5}} dx = ?$

Solution Let $u^2 = x^2 + 5$ and $2u \cdot du = 2x \cdot dx$, i.e. $xdx = udu$.

$$\text{Then } \int \frac{x}{\sqrt{x^2+5}} dx = \int \frac{u\, du}{\sqrt{u^2}} = \int du = u + c = \sqrt{x^2+5} + c.$$

EXAMPLE 80 $\int \frac{4x}{\sqrt{x-1}} dx = ?$

Solution Let $u^2 = x - 1$, $(x > 1)$ so $2u\, du = dx$.

Substituting gives $\int \frac{4x \cdot 2u}{u}\, du$, so we need to eliminate x.

We can do this by writing x in terms of u, i.e. $x = u^2 + 1$ (from the substitution $u^2 = x - 1$).

$$\text{Now, } \int \frac{4x}{\sqrt{x-1}} dx = \int \frac{4 \cdot (u^2+1)}{\sqrt{u^2}} \cdot 2 \cdot u\, du = \int 8(u^2+1)\, du$$

$$= 8 \cdot (\frac{u^3}{3} + u) + c = 8 \cdot \left(\frac{(x-1)^{\frac{3}{2}}}{3} + (x-1)^{\frac{1}{2}} \right) + c.$$

Check Yourself 10

Evaluate the integrals.

a. $\int \sqrt{1+4x}\ dx$ b. $\int \sqrt[5]{4x-3}\ dx$ c. $\int \dfrac{5x}{\sqrt{2x^2+3}}\ dx$

d. $\int \dfrac{x^2}{\sqrt{x^3-2}}\ dx$ e. $\int \dfrac{3x}{\sqrt{x-1}}\ dx$ f. $\int \dfrac{\sqrt{x+1}+1}{\sqrt{x+1}}\ dx$

Answers

a. $\dfrac{(4x+1)^{3/2}}{6}+c$ b. $\dfrac{5(4x-3)^{6/5}}{24}+c$ c. $\dfrac{5\sqrt{2x^2+3}}{2}+c$

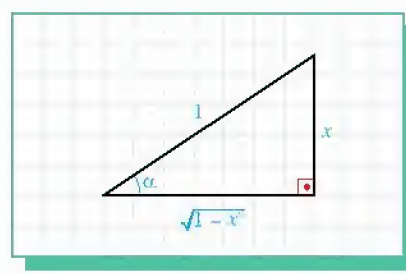
d. $\dfrac{2\sqrt{x^3-2}}{3}+c$ e. $2(x+2)\sqrt{x-1}+c$ f. $2\sqrt{x+1}+x+c$

2. Integrals of the Form $\int \sqrt{a^2 \pm u^2}\ dx$ or $\int \sqrt{u^2 \pm a^2}\ dx$

We can evaluate integrals of this kind by using trigonometric substitution.

We begin by drawing a right triangle and labeling the sides a, u, and $\sqrt{a^2 \pm u^2}$ or $\sqrt{u^2 \pm a^2}$, then we integrate the resulting trigonometric expression.

EXAMPLE 81 $\int \sqrt{1-x^2}\ dx = ?$



Solution Look at the figure.

Let $\sin\alpha = x$, then

$\cos\alpha\ d\alpha = dx$,

$\alpha = \arcsin x$.

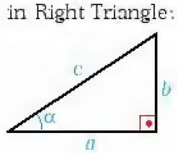
Then $\int \sqrt{1-x^2}\ dx = \int \sqrt{1-\sin^2\alpha}\cos\alpha\ d\alpha$

$$= \int \cos^2\alpha\ d\alpha = \int \frac{1+\cos 2\alpha}{2} d\alpha$$

$$= \frac{1}{2}\int d\alpha + \frac{1}{2}\int \cos 2\alpha\ d\alpha = \frac{\alpha}{2} + \frac{1}{2}\cdot\frac{1}{2}\cdot\sin 2\alpha + c$$

$$= \frac{\alpha}{2} + \frac{1}{4}\cdot 2\cdot\sin\alpha\cdot\cos\alpha + c = \frac{\arcsin x}{2} + \frac{1}{2}\cdot x\cdot\sqrt{1-x^2} + c.$$

Trigonometric Ratios in Right Triangle:



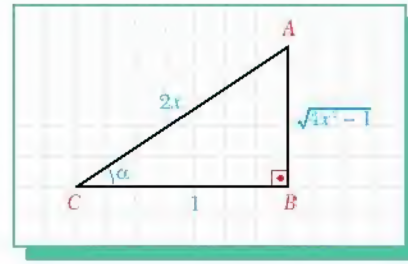
$\cos\alpha = \dfrac{a}{c}$

$\sin\alpha = \dfrac{b}{c}$

$\tan\alpha = \dfrac{b}{a}$

EXAMPLE 82 $\int \frac{\sqrt{4x^2-1}}{x}\,dx = ?$

Solution Look at the figure. Let $2x = \sec\alpha$ then $2\,dx = \tan\alpha\cdot\sec\alpha\,d\alpha$ and $\alpha = \text{arcsec}\,2x$, so



$$\int \frac{\sqrt{4x^2-1}}{x}\,dx = \int \frac{\sqrt{\sec^2\alpha-1}}{\frac{\sec\alpha}{2}}\cdot\frac{\tan\alpha\sec\alpha}{2}\,d\alpha$$

$$= \int \tan\alpha\cdot\tan\alpha\,d\alpha = \int \tan^2\alpha\,d\alpha$$

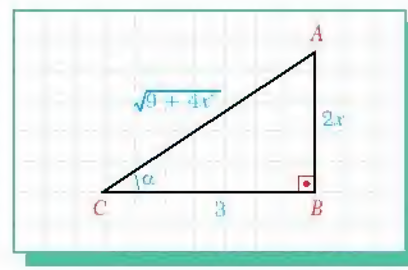
$$= \int (\tan^2\alpha+1-1)\,d\alpha$$

$$= \int (\tan^2\alpha+1)\,d\alpha - \int d\alpha$$

$$= \tan\alpha-\alpha+c = \sqrt{4x^2-1}-\text{arcsec}\,2x+c$$

EXAMPLE 83 $\int \frac{1}{x^2\cdot\sqrt{9+4x^2}}\,dx = ?$

Solution Let $\frac{2x}{3} = \tan\alpha$, so



$$x = \frac{3}{2}\tan\alpha,$$

$$dx = \frac{3d\alpha}{2\cos^2\alpha}.\ \text{Then}$$

$$\int \frac{dx}{x^2\cdot\sqrt{9+4x^2}} = \int \frac{3d\alpha}{2\cdot\cos^2\alpha\cdot\frac{9}{4}\cdot\tan^2\alpha\cdot\sqrt{9+4\cdot\frac{9}{4}\cdot\tan^2\alpha}}$$

$$= \frac{3d\alpha}{\frac{9}{2}\cos^2\alpha\cdot\tan^2\alpha\cdot 3\cdot\sqrt{1+\tan^2\alpha}}$$

$$= \int \frac{2d\alpha}{9\cos^2\alpha\cdot\frac{\sin^2\alpha}{\cos^2\alpha}\cdot\frac{1}{\cos\alpha}} = \frac{2}{9}\int \frac{\cos\alpha}{\sin^2\alpha}\,d\alpha.$$

Now use the substitutions $u = \sin\alpha$ and $du = \cos\alpha\,d\alpha$:

$$\frac{2}{9}\int \frac{\cos\alpha}{\sin^2\alpha}\,d\alpha = \frac{2}{9}\int \frac{du}{u^2} = -\frac{2}{9u}+c = -\frac{2}{9\sin\alpha}+c$$

$$= -\frac{2\cdot\sqrt{9+4x^2}}{9\cdot 2x}+c = -\frac{\sqrt{9+4x^2}}{9x}+c.$$

Check Yourself 11

Evaluate the integrals.

a. $\int \sqrt{9-x^2}\, dx$ b. $\int \frac{1}{\sqrt{9-x^2}}\, dx$ c. $\int \frac{x}{\sqrt{9x^2+4}}\, dx$

Answers

a. $\frac{9}{2}\arcsin(\frac{x}{3}) + \frac{x\sqrt{9-x^2}}{2} + c$ b. $\arcsin(\frac{x}{3}) + c$ c. $\frac{\sqrt{9x^2+4}}{9} + c$

E. INTEGRATING TRIGONOMETRIC FUNCTIONS

Let us now turn our attention to methods for evaluating the integral of complex trigonometric expressions. In this section, we will use the following basic identities:

$$\int \sin x\, dx = -\cos x + c$$

$$\int \cos x\, dx = \sin x + c$$

$$\int \tan x\, dx = \int \frac{\sin x}{\cos x}\, dx = -\ln|\cos x| + c = \ln\left|\frac{1}{\cos x}\right| + c.$$

1. Integrals of the Form $\int \sin^m x \cdot \cos^n x\, dx \quad (m, n \in \mathbb{N})$

Case 1: m and n are both odd numbers

Let $m = 2k + 1$ and $n = 2t + 1$, then we can write the integral as $\int \sin^{2k+1} x \cdot \cos^{2t} x \cdot \cos x\, dx$. Using the substitutions $u = \sin x$, $du = \cos x\, dx$ and $\cos^{2t} x = (1 - \sin^2 x)^t$ we can evaluate the integral.

Alternatively we can write $\int \cos^{2t+1} x \cdot \sin^{2k} x \cdot \sin x\, dx$ and use the substitutions $u = \cos x$, $du = -\sin x\, dx$ and $\sin^{2k} x = (1-\cos^2 x)^k$.

EXAMPLE 84 Evaluate $\int \cos^7 x \cdot \sin^3 x\, dx$.

Solution $\int \cos^7 x \cdot \sin^3 x\, dx = \int \cos^7 x \cdot \sin^2 x \cdot \sin x\, dx = \int \cos^7 x \cdot (1 - \cos^2 x) \sin x\, dx$

Use the substitutions $u = \cos x$ and $du = -\sin x\, dx$, then

$$\int \cos^7 x \cdot \sin^3 x\, dx = -\int u^7(1-u^2)\, du = \int (u^9 - u^7)\, du = \frac{u^{10}}{10} - \frac{u^8}{8} + c = \frac{\cos^{10} x}{10} - \frac{\cos^8 x}{8} + c.$$

Case 2: one of m or n is odd

In this situation we reduce the odd power by one by writing, for example,

$$\sin^7 x = \sin^6 x \cdot \sin x, \text{ or } \cos^3 x = \cos^2 x \cdot \cos x.$$

Then we can use the substitution $u = \sin x$ or $u = \cos x$ to evaluate the integral as described in Case 1.

EXAMPLE 85 Evaluate $\int \sin^6 x \cdot \cos^3 x \, dx$.

Solution $\int \sin^6 x \cdot \cos^3 x \, dx = \int \sin^6 x \cdot \cos^2 x \cdot \cos x \, dx = \int \sin^6 x \cdot (1 - \sin^2 x) \cdot \cos x \, dx$

Use the substitutions $u = \sin x$ and $du = \cos x \, dx$:

$$\int \sin^6 x \cdot \cos^3 x \, dx = \int u^6 \cdot (1 - u^2) \, du = \int (u^6 - u^8) \, du = \frac{u^7}{7} - \frac{u^9}{9} + c = \frac{\sin^7 x}{7} - \frac{\sin^9 x}{9} + c.$$

Case 3: m and n are both even numbers

In this situation, we use the following identities to evaluate the integral.

$$\sin^2 x = \frac{1 - \cos 2x}{2}$$

$$\cos^2 x = \frac{1 + \cos 2x}{2}$$

EXAMPLE 86 Evaluate $\int \sin^2 x \cdot \cos^2 x \, dx$.

Solution $\int \sin^2 x \cdot \cos^2 x \, dx = \int \frac{1 - \cos 2x}{2} \cdot \frac{1 + \cos 2x}{2} \, dx = \frac{1}{4} \int (1 - \cos^2 2x) \, dx$

$$= \frac{1}{4} \int \sin^2 2x \, dx = \frac{1}{4} \int \frac{1 - \cos 4x}{2} \, dx = \frac{1}{8} \int dx - \frac{1}{8} \int \cos 4x \, dx$$

$$= \frac{x}{8} - \frac{1}{8} \cdot \frac{1}{4} \cdot \sin 4x + c = \frac{x}{8} - \frac{\sin 4x}{32} + c.$$

2. Integrals of the Form $\int \sin mx \cdot \cos nx\, dx$, $\int \sin mx \cdot \sin nx\, dx$ or $\int \cos mx \cdot \cos nx\, dx$

To evaluate these types of integral we use the following inverse conversion identities:

$$\sin a \cdot \sin b = -\frac{1}{2}[\cos(a+b) - \cos(a-b)]$$

$$\sin a \cdot \cos b = \frac{1}{2}[\sin(a+b) + \sin(a-b)]$$

$$\cos a \cdot \cos b = \frac{1}{2}[\cos(a+b) + \cos(a-b)].$$

EXAMPLE 87 $\int \sin 3x \cdot \cos 9x\, dx = ?$

Solution $\int \sin 3x \cdot \cos 9x\, dx = \int \frac{1}{2}[\sin 12x + \sin(-6x)]\, dx = \frac{1}{2}\int (\sin 12x - \sin 6x)\, dx$

$$= -\frac{1}{2} \cdot \frac{1}{12}\cos 12x + \frac{1}{2} \cdot \frac{1}{6} \cdot \cos 6x + c = -\frac{\cos 12x}{24} + \frac{\cos 6x}{12} + c$$

EXAMPLE 88 $\int \cos 6x \cdot \cos 2x\, dx = ?$

Solution $\int \cos 6x \cdot \cos 2x\, dx = \int \frac{1}{2}[\cos 8x + \cos 4x]\, dx = \frac{1}{2}\int (\cos 8x + \cos 4x)\, dx$

$$= \frac{1}{2} \cdot \frac{1}{8} \cdot \sin 8x + \frac{1}{2} \cdot \frac{1}{4} \cdot \sin 4x + c = \frac{\sin 8x}{16} + \frac{\sin 4x}{8} + c$$

Check Yourself 12

Evaluate the integrals.

a. $\int \sin^3 x \cdot \cos x\, dx$ b. $\int \sin^4 x \cdot \cos^5 x\, dx$ c. $\int \sin^4 x \cdot \cos^4 x\, dx$

d. $\int \cos 2x \cdot \cos x\, dx$ e. $\int \cos 4x \cdot \sin 5x\, dx$ f. $\int \sin 3x \cdot \sin 5x\, dx$

Answers

a. $\frac{\sin^4 x}{4} + c$ b. $\frac{\sin^5 x}{5} - \frac{2\sin^7 x}{7} + \frac{\sin^9 x}{9} + c$ c. $\frac{3x}{128} - \frac{\sin 4x}{128} + \frac{\sin 8x}{1024} + c$

d. $\frac{\sin 3x}{6} + \frac{\sin x}{2} + c$ e. $-\frac{\cos 9x}{18} - \frac{\cos x}{2} + c$ f. $\frac{\sin 2x}{4} - \frac{\sin 8x}{16} + c$

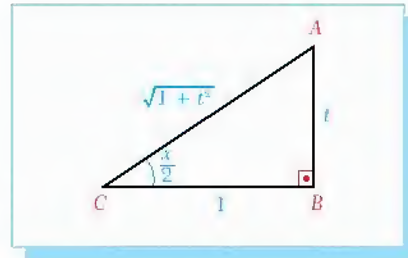
3. Substituting $t = \tan\frac{x}{2}$

This approach is possible for integrands containing only the first power of $\sin x$ and/or $\cos x$.

Look at the steps for deriving the identities provided by $t = \tan(\frac{x}{2})$:

$\tan\frac{x}{2} = t \Rightarrow x = 2\arctan t$, i.e. $dx = \frac{2}{1+t^2}\,dt$.

From the figure, $\sin\frac{x}{2} = \frac{t}{\sqrt{1+t^2}}$ and $\cos\frac{x}{2} = \frac{1}{\sqrt{1+t^2}}$.



Simplifying these expressions gives us

$$\sin x = 2\cdot\sin\frac{x}{2}\cdot\cos\frac{x}{2} = 2\cdot\frac{t}{\sqrt{1+t^2}}\cdot\frac{1}{\sqrt{1+t^2}} = \frac{2t}{1+t^2},$$

$$\cos x = \cos^2\frac{x}{2} - \sin^2\frac{x}{2} = \frac{1}{1+t^2} - \frac{t^2}{1+t^2} = \frac{1-t^2}{1+t^2}.$$

So we have the following result:

THE u=tan(x/2) SUBSTITUTION

$$\sin x = \frac{2t}{1+t^2} \qquad \cos x = \frac{1-t^2}{1+t^2} \qquad dx = \frac{2}{1+t^2}\,dt$$

EXAMPLE 89 $\int \frac{\sin x}{1+\cos x}\,dx = ?$

Solution Substitute the identities from $t = \tan(\frac{x}{2})$:

$$\int \frac{\sin x}{1+\cos x}\,dx = \int \frac{\frac{2t}{1+t^2}\cdot\frac{2\,dt}{1+t^2}}{1+\frac{1-t^2}{1+t^2}} = \int \frac{\frac{4t\,dt}{(1+t^2)(1+t^2)}}{\frac{2}{1+t^2}} = \int \frac{2t\,dt}{1+t^2}.$$

Now we can use the substitutions $u = 1 + t^2$ and $du = 2t\,dt$:

$$\int \frac{2t\,dt}{1+t^2} = \int \frac{du}{u} = \ln|u| + c = \ln(1+t^2) + c = \ln(1+\tan^2\frac{x}{2}) + c$$

EXAMPLE 90 $\int \frac{1}{\sin x+1}\,dx = ?$

Solution Substitute the identities from $t=\tan(\frac{x}{2})$:

$$\int \frac{1}{\sin x+1}\,dx = \int \frac{\frac{2\,dt}{1+t^2}}{\frac{2t}{1+t^2}+1} = \int \frac{\frac{2\,dt}{1+t^2}}{\frac{t^2+2t+1}{1+t^2}} = \int \frac{2\,dt}{(t+1)^2}$$

Substituting $u = 1 + t$ and $du = dt$ gives

$$\int \frac{2dt}{(t+1)^2} = \int \frac{2du}{u^2} = 2\int u^{-2}du = -\frac{2}{u}+c = -\frac{2}{1+t}+c = -\frac{2}{1+\tan\frac{x}{2}}+c.$$

EXAMPLE 91 $\int \frac{dx}{1+\sin x-\cos x} = ?$

Solution Substitute the identities from $t=\tan(\frac{x}{2})$:

$$\int \frac{dx}{1+\sin x-\cos x} = \int \frac{\frac{2\,dt}{1+t^2}}{1+\frac{2t}{1+t^2}-\frac{1-t^2}{1+t^2}} = \int \frac{2\,dt}{1+t^2+2t-1+t^2} = \int \frac{2\,dt}{2t^2+2t} = \int \frac{dt}{t^2+t} = \int \frac{dt}{t(t+1)}$$

Now we can use method of partial fractions:

$$\frac{1}{t(t+1)} = \frac{A}{t}+\frac{B}{t+1} = \frac{At+A+Bt}{t(t+1)} = \frac{(A+B)t+A}{t(t+1)}$$

That gives $A = 1$ and $B = -1$.

So $\int \frac{dt}{t(t+1)} = \int \frac{dt}{t} - \int \frac{dt}{t+1} = \ln|t| + c_1 - \ln|t+1| + c_2 = \ln|\tan\frac{x}{2}| - \ln|\tan\frac{x}{2}+1| + c$

$$= \ln\left|\frac{\tan\frac{x}{2}}{\tan\frac{x}{2}+1}\right| + c.$$

Check Yourself 13

Evaluate the integrals.

a. $\int \frac{1-\sin x}{1+\cos x}\,dx$ b. $\int \frac{\cos x}{1+\sin x}\,dx$ c. $\int \frac{1-\cos x}{1+\cos x}\,dx$ d. $\int \frac{dx}{1-\sin x}$

Answers

a. $\tan\frac{x}{2} - \ln\left|\frac{2}{1+\cos x}\right| + c$ b. $\ln(\sin x + 1) + c$ c. $2\tan\frac{x}{2} - x + c$ d. $\frac{2}{1-\tan\frac{x}{2}} + c$

Gottfried Leibniz (1646-1716)

Gottfried Leibniz (pronounced 'libe-nits') was born in Leipzig (now in Germany) in 1646. His father was a professor of philosophy in Leipzig. Unfortunately, Leibniz's father died when Leibniz was six years old, so he was brought up by his mother.

At school, Gottfried learned Latin and Greek, and also studied philosophy, metaphysics and theology. When he was fourteen, he went to Leipzig University and studied philosophy and mathematics for two years before beginning to study law.

In 1672 Leibniz went to Paris and studied mathematics and physics. During this time he made some discoveries concerning the sum of a series. Leibniz visited London a year later, and became a member of the Royal Society.

When Leibniz returned to Paris he began studying calculus. At that time he was still trying to develop the calculus notation, so his papers and calculations were sometimes difficult to read. However, in 1675, Leibniz wrote a paper that used the $\int f(x)\,dx$ *notation for the first time. In the same paper he described the product rule for differentiation, and the power rule.*

At around this time, Isaac Newton sent a letter to Leibniz which explained some of Newton's results, although he did not describe his methods. Leibniz replied by describing his own results. Unfortunately, Leibniz's letter did not reach Newton for a long time, and Newton decided that Leibniz had stolen his methods. This resulted in many arguments between the two scientists. However, today we can say that Leibnitz and Newton discovered the same rules of calculus independently.

Leibniz's other important achievements in mathematics include the development of a binary system of arithmetic, and his work on determinants. In 1684 he wrote a paper on calculus whose working showed the efficiency of his integral notation. Leibniz also published papers on dynamics and philosophy.

Leibniz died on 14 November 1716 in Hannover, Germany. His findings continued to influence the work of many of the mathematicians, philosophers and physicists who followed him. Today we can see that his calculus notation and methods are just as important as Newton's methods, and for this reason, Leibniz is remembered as one of the two founding fathers of differential and integral calculus.

EXERCISES 1.2

A. Integration by Substitution

1. Evaluate the integrals using the substitution method.

a. $\int \frac{1}{x-3}\,dx$

b. $\int \frac{\cos \pi}{x}\,dx$

c. $\int \frac{x}{x^2}\,dx$

d. $\int \frac{5}{3x+1}dx$

e. $\int \sin(4x+1)\,dx$

f. $\int (1+x^2+x^3)^8 \cdot (2x+3x^2)\,dx$

g. $\int (1-x^2)^7 \cdot x\,dx$

h. $\int x \cdot \cos(x^2-5)\,dx$

i. $\int \frac{1}{\sqrt{1-16x^2}}\,dx$

j. $\int e^{\sin x} \cdot \cos x\,dx$

k. $\int \frac{\cos x}{1+\sin^2 x}\,dx$

l. $\int x \cdot \sqrt{1+x^2}\,dx$

m. $\int (x^4+x^2) \cdot (2x^3+x)\,dx$

2. Evaluate the integrals using the substitution method.

a. $\int x \cdot \cos x^2\,dx$

b. $\int x \cdot \sin(5x^2+7)\,dx$

c. $\int \frac{\ln x}{x}\,dx$

d. $\int \cot x\,dx$

e. $\int \frac{1}{(1-3x)^4}\,dx$

f. $\int \frac{e^x-e^{-x}}{e^x+e^{-x}}\,dx$

g. $\int \frac{e^x}{e^x-3}\,dx$

h. $\int \frac{\sin x}{\sqrt[5]{\cos x}}\,dx$

✪i. $\int \frac{x}{\sqrt{5x-1}}\,dx$

B. Integration by Parts

3. Evaluate the integrals using the method of integration by parts.

a. $\int e^x \cdot x\,dx$

b. $\int x^2 \cdot e^x\,dx$

c. $\int x^3 \cdot e^x\,dx$

d. $\int x \cdot \sin x\,dx$

e. $\int \frac{x^2}{e^x}\,dx$

f. $\int \arccos x\,dx$

g. $\int \ln(x+5)\,dx$

h. $\int \log x\,dx$

i. $\int \operatorname{arccot} x\,dx$

✪j. $\int \cos(\ln x)\,dx$

✪✪k. $\int \sin^2 x \cdot e^{2x}\,dx$

C. Integrating Partial Fractions

4. Evaluate the integrals by using partial fractions.

a. $\int \frac{6}{2x+1}\,dx$

b. $\int \frac{9}{(3x+1)^4}\,dx$

c. $\int \frac{x^4+2x^2+x}{x^3}\,dx$

d. $\int \frac{2x+1}{x^2+x-1}\,dx$

e. $\int \frac{7x-6}{3x^2-4x+1}\,dx$

f. $\int \frac{4x^2+5x+4}{x^2+1}\,dx$

g. $\int \frac{1}{x^2+4x+4}dx$

h. $\int \frac{1}{(x+1)^3}dx$

i. $\int \frac{3x+1}{(x+1)(x+2)}\,dx$

j. $\int \frac{x-1}{x^2-2x-3}\,dx$

k. $\int \frac{2x-1}{x^2-1}\,dx$

l. $\int \frac{1}{x^2+8x+15}dx$

m. $\int \frac{x^2+3x-1}{(x-1)^3}\,dx$

n. $\int \frac{1-x}{(1+x)^2}\,dx$

o. $\int \frac{11x^2+5x+12}{(x-1)(x^2+x+2)}\,dx$

✪p. $\int \frac{x^2+2x+2}{x^3-1}\,dx$

✪q. $\int \frac{3x^4-16x^3+19x^2-5x-4}{x^3-4x^2+1}\,dx$

✪r. $\int \frac{2x+3}{(x+1)(x^2+2)}\,dx$

✪s. $\int \frac{dx}{1-4x+x^2}$

✪t. $\int \frac{x\,dx}{3+x^4}$

D. Integrating Radical Functions

5. Evaluate the integral of each radical function.

a. $\int \sqrt{5x-1}\,dx$

b. $\int \sqrt{1-x}\,dx$

c. $\int x\cdot\sqrt{1+x^2}\,dx$

d. $\int \sqrt[3]{x+1}\,dx$

e. $\int \frac{x}{\sqrt{1-x^2}}\,dx$

f. $\int \frac{5x}{\sqrt{5x^2+3}}\,dx$

g. $\int \sqrt[5]{1+x}\,dx$

h. $\int \frac{x^2}{\sqrt[3]{1+x^3}}\,dx$

i. $\int \frac{\sqrt{x-2}+3}{\sqrt{x-2}}\,dx$

6. Evaluate the integral of each radical function.

a. $\int \sqrt{1-4x^2}\,dx$

b. $\int \frac{dx}{\sqrt{1-x^2}}$

c. $\int \frac{1}{\sqrt{1+x^2}}\,dx$

d. $\int \frac{x}{\sqrt{16x^2+1}}\,dx$

✪e. $\int \frac{\sqrt{16x^2-9}}{x}\,dx$

f. $\int \sqrt{16-9x^2}\,dx$

✪g. $\int \sqrt{9x^2+1}\,dx$

✪h. $\int \sqrt{x^2-9}\,dx$

✪i. $\int \frac{dx}{\sqrt{x^2-9}}$

E. Integrating Trigonometric Functions

7. Evaluate the integral of each trigonometric function.

a. $\int \sin^2 x\cdot\cos x\,dx$

b. $\int \sin x\cdot\cos x\,dx$

c. $\int \cos^3 x\cdot\sin^5 x\,dx$

d. $\int \cos^2 x\cdot\sin x\,dx$

e. $\int \sin^3 x\cdot\cos^5 x\,dx$

f. $\int \sin^4 x\cdot\cos^3 x\,dx$

g. $\int \cos^4 x\cdot\sin^3 x\,dx$

h. $\int \cos 4x\cdot\cos 3x\,dx$

i. $\int \sin^5 x\cdot\cos^7 x\,dx$

j. $\int \sin 3x\cdot\cos 4x\,dx$

k. $\int \sin 7x\cdot\sin 5x\,dx$

l. $\int \sin 3x\cdot\cos 8x\,dx$

m. $\int \cos 2x\cdot\sin 4x\,dx$

n. $\int \cos 5x\cdot\sin x\,dx$

o. $\int \cos x\cdot\cos 4x\,dx$

✪p. $\int \sin^6 x\cdot\cos^6 x\,dx$

8. Evaluate the integral of each function by using the substitution $t=\tan\frac{x}{2}$.

a. $\int \frac{\sin x}{3+\cos x}\,dx$

b. $\int \frac{1}{\tan x+\sin x}\,dx$

c. $\int \frac{1}{\sin x}\,dx$

d. $\int \frac{3\cdot\sin x}{1-\cos x}\,dx$

✪e. $\int \frac{1+\sin x}{1-\sin x}\,dx$

CHAPTER SUMMARY

- If $F(x)$ is a function such that $F'(x) = f(x)$ then $F(x)$ is called the primitive of the function $f(x)$ and the expression $F(x) + c$ is called the indefinite integral of $f(x)$.
- In every indefinite integral we must use the constant of integration.
- **Properties of the Indefinite Integral**
  1. $d\int f(x)\,dx = f(x)\,dx$
  2. $\frac{d}{dx}\int f(x)\,dx = \int \frac{d}{dx}f(x)\,dx = f(x)$
  3. $\int dF(x) = F(x)+c$
  4. $\int a\cdot f(x)\,dx = a\cdot\int f(x)\,dx$ for $a\in R$
  5. $\int [f(x)\pm g(x)]\,dx = \int f(x)\,dx \pm \int g(x)\,dx$
- **Basic Integration Formulas**
  1. a. $\int x^n dx = \frac{x^{n+1}}{n+1}+c,\ n\neq -1$

     b. $\int a\,dx = ax+c$ for $a\in R$
  2. a. $\int \frac{1}{x}\,dx = \ln|x|+c$

     b. $\int \frac{u'(x)\,dx}{u(x)} = \ln|u(x)|+c$
  3. a. $\int e^x\,dx = e^x+c$

     b. $\int a^x\,dx = \frac{a^x}{\ln a}+c$

     c. $\int a^{u(x)}\,dx = \frac{a^{u(x)}}{u'(x)\cdot\ln a}+c$
  4. a. $\int \sin x\,dx = -\cos x+c$

     b. $\int \cos x\,dx = \sin x+c$

     c. $\int \frac{1}{\cos^2 x}\,dx = \int \sec^2 x\,dx = \int (1+\tan^2 x)\,dx = \tan x+c$

     d. $\int \frac{1}{\sin^2 x}dx = \int \csc^2 x\,dx = \int (1+\cot^2 x)\,dx = -\cot x+c$

     e. $\int \frac{1}{\sqrt{1-x^2}}\,dx = \arcsin x+c_1 = -\arccos x+c_2$

     f. $\int \frac{1}{1+x^2}\,dx = \arctan x+c_1 = -\text{arccot}\,x+c_2$
- **Integration Methods**
  1. Integration by substitution:

     Let $f(u)$ and $u(x)$ be two functions which are differentiable with respect to u and x, respectively.

     Then $\int f(u(x))\cdot u'(x)dx = F(u(x))+c.$
  2. Integration by parts:

     Let $u = f(x)$ and $v = g(x)$ be two differentiable functions with respect to x, then $\int u\cdot v'\,dx = u\cdot v - \int v\cdot u'\,dx.$
  3. Integrating partial fractions:

     a. $\int \frac{P(x)}{Q(x)}dx$ with $\deg(P(x)) = \deg(Q(x)) - 1$

     For integrals of this type, use the substitution $u = Q(x)$ and try to find du in terms of $P(x)dx$. After this, try to find the answer.

     b. $\int \frac{P(x)}{Q(x)}\,dx$ with $\deg(P(x)) < \deg(Q(x))$ and $Q(x)$ reducible in R

     In this case, if $Q(x)$ is linear (degree 1) then we can evaluate the integral easily using the formula $\int \frac{u'}{u}\,dx = \ln|u|+c$. However, if $\deg(Q(x)) > 1$ we begin by trying to write the given expression as the sum or difference of two or more partial fractions. The rules for doing this are given below.

     - $\frac{P(x)}{(ax+b)\cdot(cx+d)} = \frac{A}{ax+b}+\frac{B}{cx+d}$
     - $\frac{P(x)}{(ax+b)^n} = \frac{A}{ax+b}+\frac{B}{(ax+b)^2}+\ldots+\frac{C}{(ax+b)^n}$
     - $\frac{P(x)}{(ax+b)\cdot(cx^2+dx+e)} = \frac{A}{ax+b}+\frac{Bx+C}{cx^2+dx+e}$

     c. $\int \frac{P(x)}{Q(x)}dx$ with $\deg(P(x)) < \deg(Q(x))$ and $Q(x)$ not reducible in R.

     Given $\int \frac{dx}{ax^2+bx+c}$, if $\Delta = b^2-4ac < 0$ then use:

$$\int \frac{dx}{ax^2+bx+c} = \frac{1}{r^2}\int \frac{dx}{\left(\frac{mx+n}{r}\right)^2+1}$$

$$= \frac{1}{rm}\arctan\left(\frac{mx+n}{r}\right)+c$$

d $\int \frac{P(x)}{Q(x)}dx$ with $\deg(P(x)) \geq \deg(Q(x))$

In this situation, first divide the numerator by the denominator then calculate the integrals separately.

4 Integrating radical functions:

a. Simple radical functions:

To integrate this type of function, try to eliminate the radical sign. To do this, we use substitutions such as u^2, u^3, etc. depending on the degree of the root.

b. Integrals of the form $\int \sqrt{a^2 \pm u^2}\, dx$ or $\int \sqrt{u^2 \pm a^2}\, dx$

Use trigonometric substitution. Begin by drawing a right triangle and labeling the sides as a, u and $\sqrt{a^2 \pm u^2}$ or $\sqrt{u^2 \pm a^2}$, then integrate the resulting trigonometric expression.

5 Integrating trigonometric functions:

a. Integrals of the form $\int \sin^m x \cdot \cos^n x\, dx$

➢ If m and n are both odd numbers: First write the given expression as $\int \sin^{2m+1} x \cdot \cos^{2n} x \cdot \cos x\, dx$ then use the substitutions $u = \sin x$ and $\cos^{2n} x = (\frac{1+\cos 2x}{2})^n$ to evaluate the integral.

➢ If one of m and n is odd:
Change the term with an odd power to an equivalent term with an even power, then use the substitution $u = \sin x$ or $u = \cos x$ to evaluate the integral.

➢ If m and n are both even:
Use the identities $\sin^2 x = \frac{1-\cos 2x}{2}$ and $\cos^2 x = \frac{1+\cos 2x}{2}$ to evaluate the integral.

b. Integrals of the form $\int \sin mx \cdot \cos nx\, dx$, $\int \sin mx \cdot \sin nx\, dx$, or $\int \cos mx \cdot \cos nx\, dx$

To solve this type of integral, use the inverse conversion formulas:

$$\sin a \cdot \sin b = -\frac{1}{2}[\cos(a+b) - \cos(a-b)]$$

$$\sin a \cdot \cos b = \frac{1}{2}[\sin(a+b) + \sin(a-b)]$$

$$\cos a \cdot \cos b = \frac{1}{2}[\cos(a+b) + \cos(a-b)]$$

c. Integrals containing first only the first power of $\sin x$ and/or $\cos x$.

Use the substitution $t = \tan\frac{x}{2}$ as result of which we have:

$$\sin x = \frac{2t}{1+t^2},\ \cos x = \frac{1-t^2}{1+t^2},\ dx = \frac{2dt}{1+t^2}$$

Concept Check

- What is the relation between the derivative and the integral of a function?
- What is the primitive of a function?
- Why do we need to add a constant c when calculating the indefinite integral of a function?
- Why do we need to write 'dx' when we integrate a function?
- Can we integrate a function with respect to a variable other than x?
- What are the properties of the indefinite integral?
- Is there a function whose integral is the same function?
- Can we directly integrate trigonometric functions?
- Can we get different answers when we integrate a function by two different methods? If so, which answer is correct?
- What are the main methods of integration?
- What is the substitution method of integration?
- In the method of integration by parts, what is a good rule for selecting u and v'?
- Describe a method for integrating a rational function if the degree of the numerator is greater than the degree of the denominator.
- How can we use trigonometric substitution to integrate a radical function?
- Describe a method for integrating a trigonometric function of the form $\int \sin a^m x \cdot \cos^n x\, dx$ if m and n are both even.
- How can the substitution $t = \frac{\tan x}{2}$ help to evaluate an integral?
- If we cannot integrate a function by using the basic integration formulas, which rules we can use?

CHAPTER REVIEW TEST 1A

1. If $f(x)=\int (x^2-x+3)\,dx$ then what is $f'(2)$?

A) 1 B) 3 C) $\frac{8}{3}$ D) $\frac{23}{3}$ E) 5

2. $\int (3x^2+4x-5)\,dx = ?$

A) x^3+x^2-5x B) x^3+2x^2-5x+c

C) $3x^3+4x^2-5x+c$ D) $6x-4+c$

E) $3x^3-4x^2-5x+c$

3. $f(x)=\int \frac{x+x\sqrt{x}-x^2}{\sqrt{x}}\,dx$ is given. Find $f(4)$ if the constant of the integration is 0.

A) 1 B) 2 C) $\frac{9}{2}$ D) $\frac{17}{3}$ E) $\frac{8}{15}$

4. What is the primitive of the function

$f(x)=5x^2+3x-\frac{2}{x}$?

A) $\frac{5x^3}{3}+\frac{3x^3}{2}+\frac{2}{x}+c$

B) $\frac{4x^3}{3}+\frac{3x^2}{2}-\frac{2}{x^2}+c$

C) $\frac{5x^3}{3}+\frac{3x^2}{2}-2\ln|x|+c$

D) $10x+3+\frac{2}{x^2}+c$

E) $5x^2+3x-\frac{2}{x}+c$

5. $f'(x)=3x^2+2x+4$ and $f(1)=3$ are given. What is $f(3)$?

A) 7 B) 6 C) $\frac{1}{3}$ D) 27 E) 45

6. What is the primitive of the function $f(x)=3\cos x-4\sin x$?

A) $3\sin x+4\cos x+c$

B) $3\sin x-4\cos x+c$

C) $-3\sin x-4\cos x+c$

D) $-3\sin x+4\cos x+c$

E) $\frac{3\cos^2 x}{2}-2\sin 2x+c$

7. $\int \frac{3}{2+x}\,dx = ?$

A) $\frac{2}{3}\ln|x|+c$ B) $3\ln|2+x|+c$

C) $\frac{1}{3}\ln|2+x|+c$ D) $\frac{1}{2}\ln\frac{3}{x}+c$

E) $\frac{3}{\ln|2+x|}+c$

8. $\int (\cot^2 x+1)\,dx = ?$

A) $-\cot x+c$ B) $\cot x+c$

C) $\sin x+c$ D) $\tan x+c$

E) $\frac{\cot^3 x}{3}+x+c$

9. $\int \cot x \, dx = ?$

A) $\ln|\sin x| + c$
B) $\ln|\cos x| + c$
C) $-\cot 2x + c$
D) $\sec 2x + c$
E) $-\csc 2x + c$

10. $\int \arccos x \, dx = ?$

A) $x \arccos x + x + c$
B) $x \arccos x + \sqrt{1-x^2} + c$
C) $\arccos x + x \arccos x + c$
D) $x \arccos x - \sqrt{1-x^2} + c$
E) $x \arccos x + \sqrt{1+x^2} + c$

11. $\int \dfrac{5}{3x+1} \, dx = ?$

A) $5\ln|3x+1| + c$
B) $\dfrac{5}{3}\ln|3x+1| + c$
C) $15\ln|3x+1| + c$
D) $\ln|3x+1| + c$
E) $3\ln|3x+1| + c$

12. $\int (\cos 2x - 3) \, dx = ?$

A) $\dfrac{\sin 2x}{2} + c$
B) $\dfrac{\sin 2x}{2} - 3x + c$
C) $2\sin x + c$
D) $2\cos x + c$
E) $\dfrac{\cos 2x}{2} + x +$

13. $\int 9 \cdot \sin^9 x \cdot \cos x \, dx = ?$

A) $\dfrac{\sin^{10} x}{10} + c$
B) $\dfrac{\cos^{10} x}{10} + c$
C) $\dfrac{9\sin^{10} x}{10} + c$
D) $\dfrac{9\cos^{10} x}{10} + c$
E) $\dfrac{9\sin^{10} x \cdot \cos^2 x}{20} + c$

14. $\int \dfrac{4x-1}{2x^2-x+5} \, dx = ?$

A) $3 \cdot \ln(x-5) + c$
B) $2 \cdot \ln|x+2| + 3 \cdot \ln|x-5| + c$
C) $\dfrac{1}{4x+1} + c$
D) $\dfrac{4}{4x-1} + c$
E) $\ln(2x^2 - x + 5) + c$

15. $\int \cos 4x \cdot \cos 2x \, dx = ?$

A) $\dfrac{\cos 4x}{4} + \dfrac{\cos 2x}{2} + c$
B) $\dfrac{\sin 4x \cdot \sin 2x}{8} + c$
C) $\dfrac{\cos 6x}{12} + \dfrac{\cos 2x}{4} + c$
D) $\dfrac{\sin 6x}{12} + \dfrac{\sin 2x}{4} + c$
E) $\dfrac{\cos x}{2} + \dfrac{\sin x}{3} + c$

16. $\int \cos^2 x \, dx = ?$

A) $\dfrac{\sin^2 x}{2} + c$
B) $\dfrac{x}{2} + \dfrac{\sin 2x}{4} + c$
C) $\dfrac{\cos^3 x}{3} + c$
D) $\cos x + c$
E) $\dfrac{x}{2} + \dfrac{\cos 2x}{4} + c$

1. $f(x)=\int d(x+1)$ and $f(1)=2$ are given. What is $f(2)$?

A) 0 B) 1 C) 2 D) 3 E) –2

2. $\int \frac{x^3+4x^2-3x}{x^2}\,dx = ?$

A) $\frac{x^2}{2}+4x-3+c$ B) $\frac{x^2}{2}+4x-3\cdot\ln|x|+c$

C) $\dfrac{\frac{x^4}{4}+\frac{4x^3}{3}-\frac{3x^2}{2}}{\frac{x^3}{3}}+c$ D) $x+4-\frac{3}{x}+c$

E) x^3+4x^2-3x

3. $\int \frac{f'(x)}{f(x)}\,dx = ?$

A) $f^2(x)+c$ B) $f(x)+c$

C) $\ln|f(x)|+c$ D) $-\ln(f^2(x))+c$

E) $f^2(x)$

4. $\int \frac{f(x)}{x^2}\,dx = x^3+4x^2+5x-1$ is given.

What is $f(1)$?

A) –7 B) 3 C) –2 D) 16 E) 20

5. $\int (x^2+4x)^5\cdot(x+2)\,dx = ?$

A) $\frac{(x^2+4x)^6}{6}+c$ B) $\frac{(x^2+4x)^6}{12}+c$

C) $\frac{(x^2+4x)^6}{3}+c$ D) $\frac{2x+4}{6}+c$

E) $\dfrac{\left(\frac{x^3}{3}+x^2\right)^6\cdot\left(\frac{x^2}{2}+x\right)}{6}+c$

6. $\int \frac{1}{2\sqrt{x}}\,dx = ?$

A) $\frac{1}{2x}+c$ B) $\sqrt{x}+c$ C) $\frac{\sqrt{x}}{2}+c$

D) $\frac{x\sqrt{x}}{3}+c$ E) $\frac{1}{3\sqrt{x}}+c$

7. $\int \sin 3x\,dx = ?$

A) $\cos 3x+c$ B) $\frac{1}{3}\sin 3x+c$

C) $-\frac{1}{3}\cos 3x+c$ D) $\sin 3x+c$

E) 0

8. $\int \frac{d(\sin x)}{\sin x} = ?$

A) $\sin x+c$ B) $\cos x+c$

C) $\ln|\sin x|+c$ D) $-\ln|\cos x|+c$

E) $\ln|\cos x|+c$

9. $\int x \sin x \, dx = ?$

A) $x\cos x + \sin x + c$ B) $x\cos x - \sin x + c$

C) $x^2\cos x + x\sin x + c$ D) $-x\cos x + \sin x + c$

E) $x\cos x - \sin x + c$

10. $\int e^x \cos x \, dx = ?$

A) $\frac{e^x}{2}\cdot(\cos x + \sin x) + c$

B) $x\arccos x + \sqrt{1-x^2} + c$

C) $\arccos x + x\arccos x + c$

D) $\sec 2x + c$

E) $x\arccos x + \sqrt{1+x^2} + c$

11. $\int \frac{1}{x^2+2x+2}\,dx = ?$

A) $\arctan(x-1) + c$ B) $\arctan(x+1) + c$

C) $\text{arccot}(x+1) + c$ D) $\ln|x^2+2x+2| + c$

E) $2\ln|x+1| + c$

12. $\int \sin^3 x \cdot \cos^5 x \, dx = ?$

A) $15\sin^2 x\cos^4 x + c$ B) $8\sin^2 x + 8\cos^4 x + c$

C) $\frac{\sin^4 x}{4} + \frac{\cos^5 x}{5} + c$ D) $\frac{\cos^8 x}{8} - \frac{\cos^6 x}{6} + c$

E) $\frac{\sin^7 x}{7} - \frac{\sin^5 x}{5} + c$

13. $\int \sin 2x \cdot \cos 4x \, dx = ?$

A) $\frac{\cos 2x}{4} - \frac{\cos 6x}{12} + c$

B) $\frac{\sin 2x - \cos 6x}{12} + c$

C) $\frac{\sin^2 2x}{2} + \frac{\cos^4 4x}{4} + c$

D) $\frac{\sin 6x}{6} - \frac{\cos 2x}{2} + c$

E) $\frac{\cos 2x - 2\sin 4x}{4} + c$

14. $\int x^2 \cdot f(x)\, dx = 5x^4 + 2x^2 - 1$ is given. What is $f(x)$?

A) $20x + \frac{4}{x}$ B) $20x^3 + 4x + c$

C) $\frac{5x^3}{3} - 2x + c$ D) $5x^2 + 2 + c$

E) $\frac{2}{x} + 5x + c$

15. $\int 4 \cdot e^{4x+4}\, dx = ?$

A) $e^{4x+4} + c$ B) $4e^{4x+1} + c$

C) $\frac{e^{4x+1}}{4} + c$ D) $4e^{4x+3} + c$

E) $e^{4x+1} + c$

16. $\int \frac{\cos(\ln x)}{x}\, dx = ?$

A) $\sin x + c$ B) $\cos x + c$

C) $\cos(\ln x) + c$ D) $\sin(\ln x) + c$

E) $\cos(\sin x) + c$

1. $f(w) = \int (xw - w)\ dw$ is given. What is $f(w)$?

A) $\dfrac{x^2w}{2} - wx + c$ B) $\dfrac{xw^2}{2} - wx + c$

C) $\dfrac{xw^2}{2} - \dfrac{w^2}{2} + c$ D) $x^2w^2 + w^2 + c$

E) $xw - \dfrac{w^2}{2} + c$

2. $f(x) = \int (x^2 + x - 2)\ dx$ and $f(1) = 2$ are given. What is $f(2)$?

A) $\dfrac{23}{6}$ B) $\dfrac{17}{3}$ C) $\dfrac{9}{2}$ D) 21 E) 13

3. $\int e^{\sin x} \cos x\ dx = ?$

A) $\sin x + c$ B) $e^{\sin x} + c$

C) $e^{\cos x} + c$ D) $\cos x + c$

E) $\dfrac{1}{e} + c$

4. $\int \dfrac{3}{\sqrt{1-9x^2}}\ dx = ?$

A) $\arcsin x + c$ B) $\arccos x + c$

C) $\arcsin 3x + c$ D) $\arctan 3x + c$

E) $\arccos 3x + c$

5. $\int e^{3x^2+4} \cdot x\ dx = ?$

A) $\dfrac{e^{3x^2+4}}{3x^2+4} + c$ B) $\dfrac{e^{6x}}{6x} + c$

C) $\dfrac{e^{3x^2+4}}{6} + c$ D) $6x \cdot e^{3x^2+4} + c$

E) $\dfrac{e^{3x^2+4}}{x} + c$

6. $\int \dfrac{x}{(x^2-1)^2}\ dx = ?$

A) $\dfrac{x^3}{3} + c$ B) $\dfrac{(x^2-1)^3}{4} + c$

C) $\ln|x^2 - 1| + c$ D) $\dfrac{-1}{2(x^2-1)} + c$

E) $\dfrac{x^2-1}{x} + c$

7. $\int 5 \cdot e^{7x-2}\ dx = ?$

A) $35e^{7x-2} + c$ B) $\dfrac{7e^{7x-2}}{5} + c$

C) $35e^7 + c$ D) $\dfrac{5e^{7x-2}}{7} + c$

E) $\dfrac{5e^7}{7} + c$

8. $\int 3e^{\sin^2 x} \sin 2x\ dx = ?$

A) $\dfrac{1}{2}\sin^2 x + c$ B) $\dfrac{1}{3}e^{\sin^2 x} + c$

C) $-\dfrac{1}{3}e^{\sin^2 x} + c$ D) $6e^{\sin^2 x} + c$

E) $3e^{\sin^2 x} + c$

9. $\int e^{3x} x^2 \; dx = ?$

A) $e^{3x} + x^3 + c$

B) $\frac{x^2 e^{3x}}{3} - \frac{2xe^{3x}}{9} + \frac{2e^{3x}}{27} + c$

C) $\frac{xe^{3x}}{3} + \frac{x^2 e^{3x}}{9} + \frac{x^3 e^{3x}}{27} + c$

D) $\frac{e^{3x}}{3} + \frac{x^3}{3} + c$

E) $\frac{e^{3x} x^2}{3} - 6xe^{3x} + e^{3x} + c$

10. $\int x^3 \ln x \; dx = ?$

A) $\frac{x^2 \ln x}{3} + \frac{x}{9} + c$

B) $\frac{x \ln x}{2} + \frac{x^2 \ln x}{4} + c$

C) $\frac{x^4 \ln x}{4} - \frac{x^4}{16} + c$

D) $\frac{x^3 \ln x}{4} + \frac{x^3}{16} + c$

E) $\frac{x^4 \ln x}{4} + \frac{x^4}{16} + c$

11. $\int \frac{x-1}{x^3+1} \; dx = ?$

A) $\ln|x^3 + 1| + c$

B) $\ln \frac{x+1}{x^2 - x + 1} + c$

C) $\ln(|x + 1|^3 |x^2 - x + 1|^2) + c$

D) $\frac{3}{4} \ln|x^3 + 1| + c$

E) $\frac{1}{3} \ln\left|\frac{x^2 - x + 1}{x^2 + 2x + 1}\right| + c$

12. $\int \frac{3x+1}{x^2 + 3x - 4} \; dx = ?$

A) $\ln|x + 4| + \ln|x - 1| + c$

B) $\ln \frac{x+4}{x-1} + c$

C) $\frac{\ln(|x+4|^{11} (x-1)^4)}{5} + c$

D) $\ln(|x + 4|^{11} (x - 1)^4) + c$

E) $\ln|x^2 + 3x - 4| + c$

13. $\int \frac{2x}{\sqrt{1 - x^2}} \; dx = ?$

A) $\ln|1 - x^2| + c$

B) $\arcsin x + c$

C) $-2\sqrt{1 - x^2} + c$

D) $2\sqrt{1 - x^2} + c$

E) $\frac{1}{2} \ln|1 - x^2| + c$

14. $\int \frac{1 - \cos x}{\sin^2 x} \, dx = ?$

A) $2\tan\frac{x}{2} + c$

B) $\tan\frac{x}{2} + c$

C) $2\tan\frac{x}{2} + 2\sin\frac{x}{2} + c$

D) $\sin x + \cos x + c$

E) $\sin\frac{x}{2} + \cos\frac{x}{2} + c$

15. $\int \cos(\cos x) \cdot \sin 2x \; dx = ?$

A) $\cos x + c$

B) $\sin(\cos x) + \cos(\sin x) + c$

C) $\sin x + c$

D) $\sin(\cos x) + \cos(\sin x) + c$

E) $-2(\cos(\cos x) + \cos x \sin(\cos x)) + c$

16. $\int e^x (x-1) \; dx = ?$

A) $e^x(x - 1) + c$

B) $e^x(x + 1) + c$

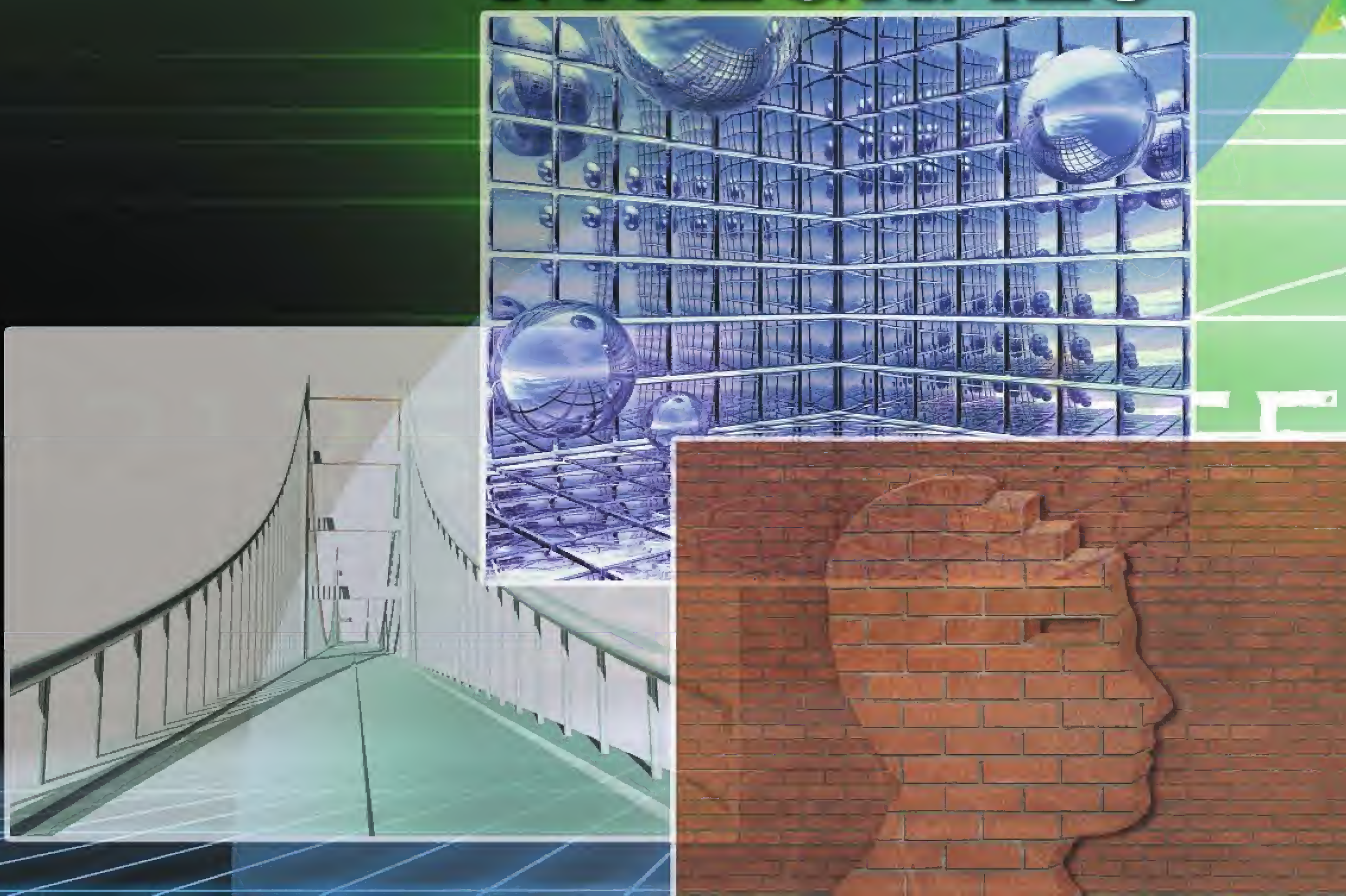
C) $xe^x + e^x + c$

D) $e^x(e^x + 1) + c$

E) $e^x(x - 2) + c$

Chapter 2

DEFINITE INTEGRALS



1 EVALUATING DEFINITE INTEGRALS

A. DEFINITION OF THE DEFINITE INTEGRAL

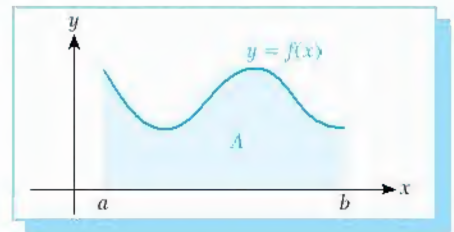
Definition

definite integral

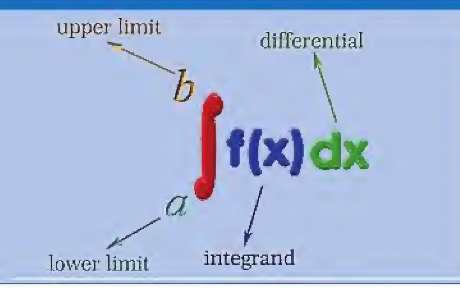
Let $f(x)$ be a continuous function defined on an interval $[a, b]$. Then the area between the graph of $f(x)$ and the x-axis is called the definite integral of $f(x)$ betwen a and b.

For example, in the figure opposite, the shaded area A shows the definite integral of $f(x)$ on the interval $[a, b]$.

We can write this expression as $A = \int_a^b f(x)\,dx$.



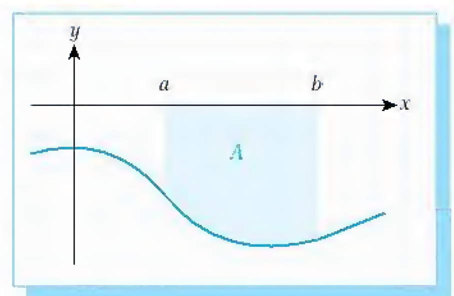
THE DEFINITE INTEGRAL



Note

If the graph is below the x-axis then its integral will be negative. However, area is a positive quantity so we reverse the sign: $A = -\int_a^b f(x)\,dx$.

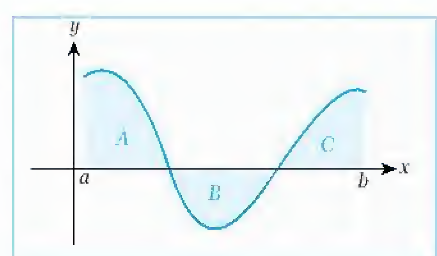
We say that the definite integral can be negative but the area is always positive.



Note

If part of the graph is below the x-axis and part of the graph is above the x-axis then the integral will be the algebraic sum of the areas.

In the figure, all of the areas A, B, C are positive numbers, so $\int_a^b f(x)\,dx = A - B + C$.



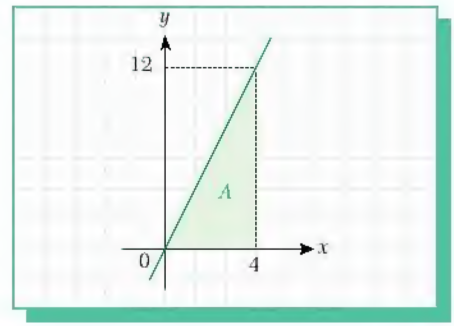
Note

For linear functions we can use geometric methods (by finding the area of a triangle) to calculate the area under a curve.

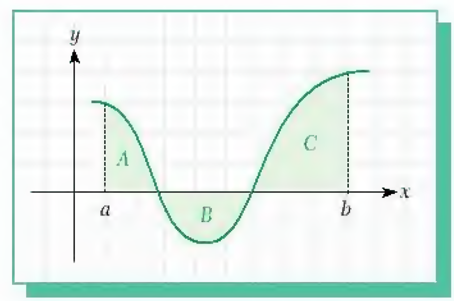
EXAMPLE 1 Find the area of the region between the graph of $y = 3x$ and the x-axis on the interval $[0, 4]$.

Solution The shaded area is

$A = \int_0^4 3x\,dx = \frac{1}{2} \cdot 4 \cdot 12 = 24$ square units, by the formula for the area of a triangle.



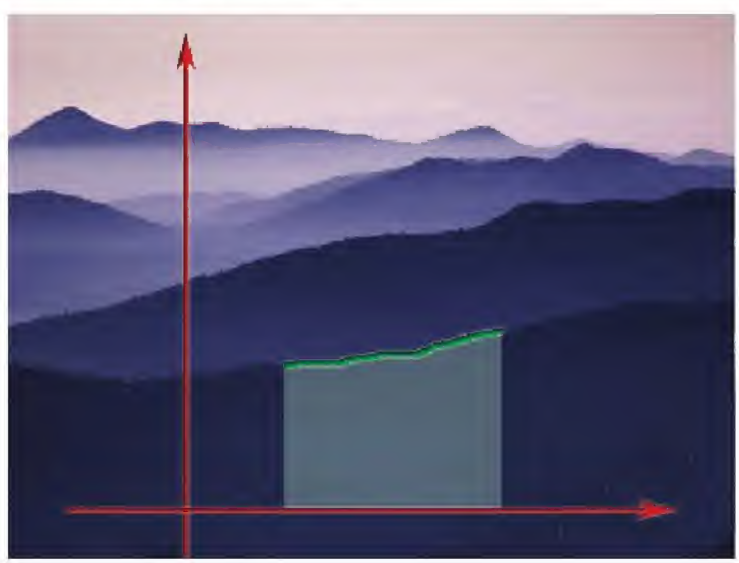
EXAMPLE 2 In the figure, the areas of the shaded parts A, B and C are given as $A = 7\text{ cm}^2$, $B = 9\text{ cm}^2$ and $C = 8\text{ cm}^2$. Find the total area of the shaded region in the figure and evaluate the integral on the interval $[a, b]$.



Solution Total area $= A + B + C = 7 + 9 + 8 = 24\text{ cm}^2$

The integral on $[a, b] = \int_a^b f(x)\,dx = A - B + C$

$= 7 - 9 + 8 = 6.$



B. THE FUNDAMENTAL THEOREM OF CALCULUS

Theorem

Fundamental Theorem of Calculus

Let $f(x)$ be a function such that $f: [a, b] \to R$. If $F'(x) = f(x)$ and $\int f(x)\, dx = F(x) + c$, then

$$\int_a^b f(x)\, dx = (F(x) + c)\Big|_a^b = F(b) - F(a)$$

Here we use the notation $\Big|_a^b$ to show the boundaries of the integral.

Note

The Fundamental Theorem of Calculus shows us that we do not need a constant of integration c when we evaluate a definite integral. For example, suppose we write $F(a) + c$ instead of $F(a)$, and $F(b) + c$ instead of $F(b)$. Then by the Fundamental Theorem of Calculus,

$$\int_a^b f(x)\, dx = (F(b)+c) - (F(a)+c) = F(b) - F(a) + c - c = F(b) - F(a).$$

FUNDAMENTAL THEOREM OF CALCULUS

$$\int_a^b f(x)\, dx = F(b) - F(a)$$

EXAMPLE 3 $\int_0^1 x\, dx = ?$

Solution $\int_0^1 x\, dx = \dfrac{x^2}{2}\Big|_0^1 = \dfrac{1^2}{2} - \dfrac{0^2}{2} = \dfrac{1}{2}.$

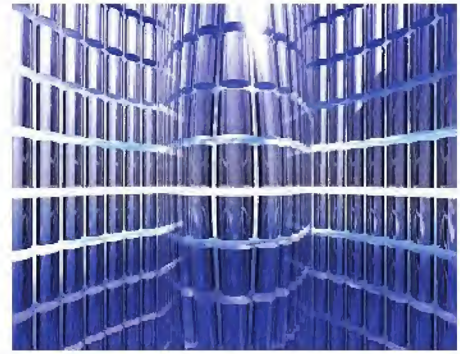
EXAMPLE 4 $\int_3^5 x^2\, dx = ?$

Solution $\int_3^5 x^2\, dx = \dfrac{x^3}{3}\Big|_3^5 = \dfrac{5^3}{3} - \dfrac{3^3}{3} = \dfrac{125-27}{3} = \dfrac{98}{3}.$

EXAMPLE 5 $\int_0^\pi \sin x\, dx = ?$

Solution $\int_0^\pi \sin x\, dx = -\cos x\Big|_0^\pi = -\cos \pi - (-\cos 0) = 1 + 1 = 2.$

EXAMPLE 6 $\int_0^1 e^{3x}\ dx = ?$



Solution $\int_0^1 e^{3x}\ dx = \frac{1}{3}e^{3x}\Big|_0^1 = \frac{e^3}{3} - \frac{e^0}{3} = \frac{e^3 - 1}{3}.$

Check Yourself 1

Evaluate the definite integrals.

a. $\int_1^5 x^3\ dx$ b. $\int_0^{\pi} \cos x\ dx$ c. $\int_0^e e^{2x}\ dx$ d. $\int_1^e \frac{1}{x}\ dx$

Answers

a. 156 b. 0 c. $\frac{e^{2e} - 1}{2}$ d. 1

C. PROPERTIES OF THE DEFINITE INTEGRAL

Let f: $|a, b| \to R$ and g: $|a, b| \to R$ be two integrable functions. Then the following properties hold:

1. $\int_a^a f(x)\ dx = 0$

Proof $\int_a^a f(x)\ dx = F(a) - F(a) = 0$

2. $\int_a^b f(x)\ dx = -\int_b^a f(x)\ dx$

Proof $\int_a^b f(x)\ dx = F(b) - F(a) = -(F(a) - F(b)) = -\int_b^a f(x)\ dx$

3. For any $c \in R$, $\int_a^b c \cdot f(x)\ dx = c \cdot \int_a^b f(x)\ dx.$

Proof $c \cdot \int_a^b f(x)\ dx = c \cdot (F(b) - F(a)) = c \cdot F(b) - c \cdot F(a) = c \cdot F(x)\Big|_a^b = \int_a^b c \cdot f(x)\ dx$

4. $\int_a^b |f(x) \pm g(x)|\ dx = \int_a^b f(x)\ dx \pm \int_a^b g(x)\ dx$

Proof $\int_a^b f(x)\ dx \pm \int_a^b g(x)\ dx = |F(b) - F(a)| \pm |G(b) - G(a)| = |F(b) \pm G(b)| - |F(a) \pm G(a)|$

$= |F \pm G|(b) - |F \pm G|(a)$

$= \int_a^b |f(x) \pm g(x)|\ dx$

5. For any a, b, $c \in R$ with $a \le b \le c$, then $\int_a^c f(x)\,dx = \int_a^b f(x)\,dx + \int_b^c f(x)$.

Proof $\int_a^c f(x)\,dx = F(c) - F(a) = F(c) - F(a) + F(b) - F(b) = |F(b) - F(a)| + |F(c) - F(b)|$

$$= \int_a^b f(x)\,dx + \int_b^c f(x)\,dx$$

6. If $f(x) \ge g(x)$ for every $x \in [a, b]$, then $\int_a^b f(x)\,dx \ge \int_a^b g(x)\,dx$.

Proof If $f(x) \ge g(x)$ then $f(x) - g(x) \ge 0$.

Let us write $h(x) = f(x) - g(x)$, so $h(x) \ge 0$ for every x on $[a, b]$.

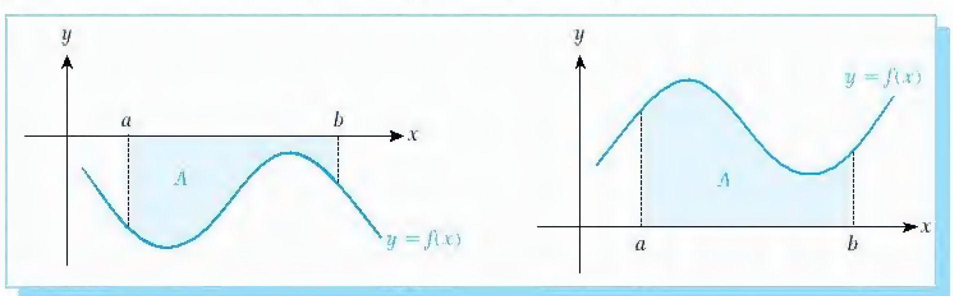
We know that $\int_a^b h(x)\,dx$ is the area between the graph of $h(x)$ and the x-axis.

If $h(x) \ge 0$ then $\int_a^b h(x)\,dx \ge 0$.

7. $\left|\int_a^b f(x)\,dx\right| \le \int_a^b |f(x)|\,dx$

Proof We know that $\int_a^b f(x)\,dx$ is the area between the graph of $f(x)$ and the x-axis on the interval $[a, b]$. So we have two possible cases:

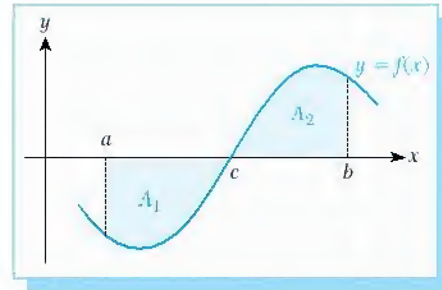
Case 1: $f(x) \le 0$, for every $x \in [a, b]$ or $f(x) \ge 0$, for every $x \in [a, b]$.



In the first figure, $A = -\int_a^b f(x)\,dx$ or $\int_a^b f(x)\,dx = -A$ and $|-A| = |A| = \int_a^b |f(x)|\,dx$.

In the second figure, $A = \int_a^b f(x)\,dx = \int_a^b |f(x)|\,dx$. This concludes half of the proof.

Case 2: Let $c \in [a, b]$. If $f(x) \le 0$ for $x \in [a, c]$ and $f(x) \ge 0$ for $x \in [c, b]$ then $\int_a^b f(x)\,dx = A_2 - A_1$ and so $\left|\int_a^b f(x)\,dx\right| = |A_2 - A_1|$.



However, $\int_a^b |f(x)|\,dx = |A_2| + |A_1|$ and

$|A_2 - A_1| \le |A_2 + A_1|$ by the triangle inequality.

So $\left|\int_a^b f(x)\,dx\right| \le \int_a^b f(x)\,dx$.

Note

1. All continuous functions have integrals on a closed interval $[a, b]$.
2. A function with a countable number of points of discontinuity has an integral on the closed interval $[a, b]$. For example, if the points $c_1, c_2, \ldots, c_n \in [a, b]$ are the points of discontinuity of $f(x)$ then $\int_a^b f(x)\,dx = \int_a^{c_1} f(x)\,dx + \int_{c_1}^{c_2} f(x)\,dx + \ldots + \int_{c_n}^{b} f(x)\,dx$.

EXAMPLE 7 $\int_3^3 x\,dx = ?$

Solution We know $\int_a^a f(x)\,dx = 0$, so $\int_3^3 x\,dx = 0$.

EXAMPLE 8 $\int_4^4 5x^2 \sin^3 4x\,dx = ?$

Solution In a similar way to the previous example we can write $\int_4^4 5x^2 \sin^3 4x\,dx = 0$.

EXAMPLE 9 $\int_1^3 (2x^2 + 3x - 4)\,dx = ?$

Solution $\int_1^3 (2x^2 + 3x - 4)\,dx = 2 \cdot \int_1^3 x^2\,dx + 3 \cdot \int_1^3 x\,dx - 4 \cdot \int_1^3 dx = 2 \cdot \frac{x^3}{3}\Big|_1^3 + 3 \cdot \frac{x^2}{2}\Big|_1^3 - 4 \cdot x\Big|_1^3$

$$= 2 \cdot (9 - \frac{1}{3}) + 3 \cdot (\frac{9}{2} - \frac{1}{2}) - 4 \cdot (3 - 1) = \frac{52}{3} + 12 - 8 = \frac{64}{3}.$$

EXAMPLE 10 $\int_1^e \left(\frac{x^2-5x+1}{x}\right) dx = ?$

Solution $\int_1^e \left(\frac{x^2-5x+1}{x}\right) dx = \int_1^e \left(x-5+\frac{1}{x}\right) dx = \int_1^e x\,dx - 5\cdot\int_1^e dx + \int_1^e \frac{1}{x}\,dx = \frac{x^2}{2}\Big|_1^e - 5x\Big|_1^e + \ln x\Big|_1^e$

$$= \left(\frac{e^2}{2}-\frac{1}{2}\right) - 5\cdot(e-1) + (\ln e - \ln 1) = \frac{e^2-10e+11}{2}.$$

EXAMPLE 11 $\int_0^1 e^{3x+1}\,dx = ?$

Solution $\int_0^1 e^{3x+1}\,dx = \frac{e^{3x+1}}{3}\Big|_0^1 = \frac{e^4-e}{3}.$

EXAMPLE 12 $\int_0^{\pi/2} (\sin x + \cos x)\,dx = ?$

Solution $\int_0^{\pi/2} (\sin x + \cos x)\,dx = (-\cos x + \sin x)\Big|_0^{\pi/2} = \left(-\cos\frac{\pi}{2} + \sin\frac{\pi}{2}\right) - (-\cos 0 + \sin 0)$

$$= (-0 + 1) - (-1 + 0) = 2.$$

EXAMPLE 13 $\int_0^1 \sqrt{3x+1}\,dx = ?$

Solution 1 We can calculate the integral using substitution:

Let $u = 3x + 1$ so $du = 3\,dx$.

Now we need to calculate the boundaries in terms of u.

The lower bound is $3 \cdot 0 + 1 = 1$.

The upper bound is $3 \cdot 1 + 1 = 4$. So we can write

$$\int_0^1 \sqrt{3x+1}\,dx = \frac{1}{3}\int_1^4 \sqrt{u}\,du = \frac{1}{3}\frac{u^{3/2}}{\frac{3}{2}}\Big|_1^4 = \frac{2u^{3/2}}{9}\Big|_1^4 = \frac{2\cdot 4^{3/2}}{9} - \frac{2\cdot 1^{3/2}}{9} = \frac{16-2}{9} = \frac{14}{9}.$$

If we calculate a definite integral using the substitution method, we must always remember to calculate the boundaries in terms of the substitution.

Solution 2 Alternatively, in this example we can calculate the integral directly:

$$\int_0^1 \sqrt{3x+1}\,dx = \frac{2}{9}(3x+1)^{3/2}\Big|_0^1 = \frac{2}{9}\cdot 4^{3/2} - \frac{2}{9}\cdot 1^{3/2} = \frac{16-2}{9} = \frac{14}{9}.$$

EXAMPLE 14 $\int_1^e x\ln x \; dx = ?$

Solution Let $u = \ln x$ and $v' = x$, then $u' = \frac{1}{x}$ and $v = \frac{x^2}{2}$.

$$\int_1^e x\ln x \, dx = \frac{x^2}{2}\ln x \Big|_1^e - \int_1^e \frac{1}{x}\cdot\frac{x^2}{2}\, dx = \left(\frac{x^2}{2}\ln x - \frac{x^2}{4}\right)\Big|_1^e = \left(\frac{e^2}{2}\ln e - \frac{e^2}{4}\right) - \left(\frac{1}{2}\ln 1 - \frac{1}{4}\right)$$

$$= \frac{e^2}{2} - \frac{e^2}{4} + \frac{1}{4} = \frac{e^2+1}{4}.$$

EXAMPLE 15 $\int_{-1}^3 \frac{x+4}{x^2+2x}\, dx = ?$

Solution We can calculate the integral using partial fractions:

$$\int_{-1}^3 \frac{x+4}{x^2+2x}\, dx = \int_{-1}^3 \frac{x+4}{x(x+2)}\, dx$$

$\frac{x+4}{x(x+2)} = \frac{A}{x} + \frac{B}{x+2} = \frac{(A+B)x+2A}{x(x+2)}$; $A+B=1$ and $2A=4$. So $A=2$ and $B=-1$, and

$$\int_{-1}^3 \frac{x+4}{x^2+2x}\, dx = \int_{-1}^3 \frac{2}{x}\, dx - \int_{-1}^3 \frac{1}{x+2}\, dx = (2\ln|x| - \ln|x+2|)\Big|_{-1}^3 = \ln\left|\frac{x^2}{x+2}\right|\Big|_{-1}^3 = \ln\frac{9}{5} - \ln 1 = \ln\frac{9}{5}.$$

Check Yourself 2

Evaluate the definite integrals.

a. $\int_2^2 x^5 \cdot \cos 4x \; dx$

b. $\int_1^4 (x^3 + 4x^2 - 3x - 2)\, dx$

c. $\int_1^e \frac{x^3+4x^2+5x-1}{x}\, dx$

d. $\int_{-2}^3 (x^2 + x - 2)\; dx$

e. $\int_{\pi/2}^{\pi} (2\cos x - \sin 2x)\; dx$

f. $\int_1^4 \frac{x-3}{2x^2+x}\, dx$

g. $\int_{-1}^3 \sqrt{2x+3}\; dx$

h. $\int_1^{e^2} \frac{\ln(\ln x)}{x}\, dx$

Answers

a. 0

b. $\frac{477}{4}$

c. $\frac{e^3}{3} + 2e^2 + 5e - \frac{25}{3}$

d. $\frac{25}{6}$

e. -1

f. $\frac{7\ln 3}{2} - 6\ln 2$

g. $\frac{26}{3}$

h. $\ln 4 - 2$

D. LEIBNIZ'S RULE

Rule

Leibniz's Rule

Let $f : [a, b] \to R$ be a continuous function such that $F(x) = \int_a^x f(t)\, dt$. Then

$$F'(x) = \frac{d}{dx}\left(\int_a^x f(t)\, dt\right) = f(x).$$

Leibniz's Rule gives us two important corollaries:

COROLLARY TO LEIBNIZ'S RULE

Let $u(x)$ and $v(x)$ be two differentiable functions. Then

1. $F(x) = \int_a^{v(x)} f(t)\, dt \Rightarrow F'(x) = f(v(x)) \cdot v'(x)$
2. $F(x) = \int_{u(x)}^{v(x)} f(t)\, dt \Rightarrow F'(x) = f(v(x)) \cdot v'(x) - f(u(x)) \cdot u'(x).$

EXAMPLE 16 $F(x) = \int_1^x \cos t^2\, dt$ is given. Find $F'(\frac{\sqrt{\pi}}{2})$.

Solution We know from corollary 1 to Leibniz's Rule that $F(x) = \int_a^{v(x)} f(t)\, dt \Rightarrow F'(x) = f(v(x)) \cdot v'(x)$.

$F(x) = \int_1^x \cos t^2\, dt$ and $f(t) = \cos t^2$, then $F'(x) = \cos x^2 \cdot (x)' = \cos x^2$.

So $F'(\frac{\sqrt{\pi}}{2}) = \cos(\frac{\sqrt{\pi}}{2})^2 = \cos\frac{\pi}{4} = \frac{\sqrt{2}}{2}$.

EXAMPLE 17 $F(x) = \int_x^{x^2} (t^2 - 4t + 1)\, dt$ is given. Find $F'(2)$.

Solution We know from corollary 2 to Leibniz's Rule that,

$$F(x) = \int_{u(x)}^{v(x)} f(t)\, dt \Rightarrow F'(x) = F(v(x)) \cdot v'(x) - f(u(x)) \cdot u'(x).$$

So if we have $F(x) = \int_x^{x^2} (t^2 - 4t + 1)\, dt$ then

$$\begin{aligned} F'(x) &= f(x^2) \cdot (x^2)' - f(x) \cdot (x)' \text{ where } f(t) = t^2 - 4t + 1, \\ F'(x) &= (x^4 - 4x^2 + 1) \cdot (2x) - (x^2 - 4x + 1) \cdot 1, \\ F'(2) &= (2^4 - 4 \cdot 2^2 + 1) \cdot (2 \cdot 2) - (2^2 - 4 \cdot 2 + 1) \\ &= 7. \end{aligned}$$

E. THE MEAN VALUE THEOREM

MEAN VALUE THEOREM

Let $f: [a, b] \to R$ be a continuous function. Then there exists at least one real number $c \in [a, b]$ such that

$$f(c) = \frac{\int_a^b f(x)\,dx}{b-a}.$$

In the given formula, $f(c)$ is called the mean value of $f(x)$ on the interval $[a, b]$.

EXAMPLE 18 Find the mean value of $f(x) = x^2 - 4x$ on the interval $[0, 4]$.

Solution Let $f(c)$ be the mean value of $f(x)$ in $[0, 4]$.

By the Mean Value Theorem we have

$$f(c) = \frac{\int_0^4 (x^2-4x)dx}{4-0}$$

$$= \frac{\left(\frac{x^3}{3} - 4\frac{x^2}{2}\right)\Big|_0^4}{4}$$

$$= \frac{\frac{64}{3} - \frac{64}{2}}{4}$$

$$= -\frac{8}{3}.$$

Check Yourself 3

1. Given $F(x) = \int_2^x \sin t^3\,dt$, find $F'(\sqrt[3]{\frac{\pi}{3}})$.

2. Given $F(x) = \int_{x^2}^{x^3} (t^2 + 4t - 1)\,dt$, find $F'(3)$.

3. Find the mean value of $f(x) = \dfrac{x^3 + 2x^2 + 1}{x^2}$ on the interval $[1, 3]$.

Answers

1. $\frac{\sqrt{3}}{2}$ 2. 21876 3. $\frac{13}{3}$

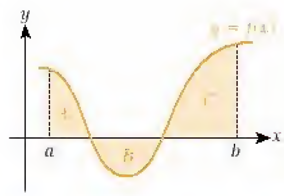
EXERCISES 2.1

A. Definition of the Definite Integral

1. Evaluate the area between the graph of $f(x) = x$ and the x-axis on the interval $[0, 3]$.

2. Find the integral of the function $f(x) = 3x - 2$ on the interval $[0, 4]$ and find the area between the graph of $f(x)$ and the the x-axis on the same interval.

3. In the figure, $A = 5\text{ cm}^2$, $B = 4\text{ cm}^2$ and $C = 7\text{ cm}^2$. Find the area of the shaded region and evaluate the integral of $f(x)$ on the interval $[a, b]$.



B. The Fundamental Theorem of Calculus

4. Evaluate the integrals.

a. $\int_2^3 x^2\,dx$

b. $\int_1^5 3x\,dx$

c. $\int_0^4 x^3\,dx$

d. $\int_1^e \frac{\ln x}{x}\,dx$

e. $\int_0^{\ln 2} e^x\,dx$

f. $\int_0^{\pi} \cos x\,dx$

g. $\int_1^4 (\sqrt{x} + \frac{1}{\sqrt{x}})\,dx$

h. $\int_0^a (ax^2 + a^2x)\,dx$

i. $\int_0^7 \sqrt{x+9}\,dx$

C. Properties of the Definite Integral

5. Evaluate the integrals.

a. $\int_2^2 (5x^2 + 4x + \cos x + e^{\ln 4})\,dx$

b. $\int_1^1 4^{\sin(\ln(\cos x))}\,dx$

c. $\int_3^4 3x^2\,dx$

d. $\int_{-3}^4 (5x^3 + 4x^2 + 3x - 5)\,dx$

e. $\int_1^4 (3x^4 + 4x^2 + \frac{1}{x} + \sqrt{x})\,dx$

f. $\int_0^{\pi} (\sin 3x + 4\cos 2x)\,dx$

g. $\int_1^e \frac{x^3 + x + 1}{x}\,dx$

6. $\int_3^5 f(x)\,dx = 7$ is given. Evaluate $\int_5^3 f(x)\,dx$.

7. $\int_1^5 f(x)\,dx = 5$ and $\int_3^5 f(x)\,dx = 8$ are given. Evaluate $\int_1^3 f(x)\,dx$.

D. Leibniz's Rule

8. a. Given $F(x) = \int_1^x \cos t\,dt$, find $\frac{dF}{dx}$.

b. Given $F(x) = \int_{1/x}^1 (t^2 + 2t)\,dt$, find $\frac{dF}{dx}$.

c. $F(x) = \int_0^{x^2} \sin t\,dt$ is given. Find $\frac{dF}{dx}$.

d. $F(x) = \int_x^{x^3} (x+4)\,dx$ is given. Find $\frac{dF}{dx}$.

E. The Mean Value Theorem

9. Find the mean value of each function on the given interval.

a. $f(x) = x + 1$ on $[0, 5]$

b. $f(x) = x^3 + 1$ on $[-1, 2]$

c. $f(x) = \frac{x^2 + 2x + 4}{x}$ on $[-3, 3]$

d. $f(x) = \sin x$ on $[0, 2\pi]$

2 INTEGRALS OF SOME SPECIAL FUNCTIONS (OPTIONAL)

A. INTEGRATING ABSOLUTE VALUE FUNCTIONS

Recall that an absolute value function $|f(x)|$ is a function such that

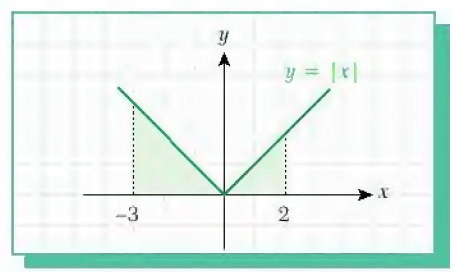
$$|f(x)| = \begin{cases} f(x), & f(x) \geq 0 \\ -f(x), & f(x) < 0. \end{cases}$$

When integrating absolute value functions, first we find the positive and negative parts of given function. For the intervals where $f(x)$ changes its sign we use the fifth property of definite integrals:

$$\text{for } a \leq b \leq c, \ \int_a^c f(x)\,dx = \int_a^b f(x)\,dx + \int_b^c f(x)\,dx.$$

EXAMPLE 19 $\int_{-3}^{2} |x|\,dx = ?$

Solution $|x| = \begin{cases} x \text{ if } x \geq 0 \\ -x \text{ if } x < 0 \end{cases}$

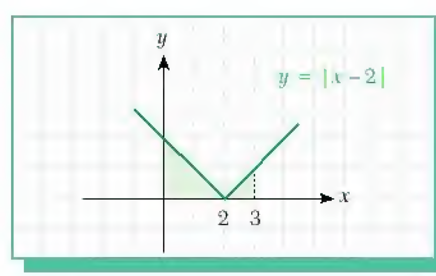


So $\int_{-3}^{2} |x|\,dx = \int_{-3}^{0} (-x)\,dx + \int_{0}^{2} x\,dx = -\frac{x^2}{2}\Big|_{-3}^{0} + \frac{x^2}{2}\Big|_{0}^{2}$

$$= \left(-\frac{0^2}{2} - \frac{-(-3)^2}{2}\right) + \left(\frac{2^2}{2} - \frac{0^2}{2}\right) = \frac{13}{2}.$$

EXAMPLE 20 $\int_{0}^{3} |x-2|\,dx = ?$

Solution $x - 2 = 0 \Rightarrow x = 2$ is the root of $x - 2$.



So $f(x) = \begin{cases} -(x-2) & \text{if } x < 2 \\ x-2 & \text{if } x \geq 2. \end{cases}$

So $\int_{0}^{3} |x-2|\,dx = \int_{0}^{2} -(x-2)\,dx + \int_{2}^{3} (x-2)\,dx$

$$= \left(-\frac{x^2}{2} + 2x\right)\Big|_{0}^{2} + \left(\frac{x^2}{2} - 2x\right)\Big|_{2}^{3}$$

$$= -2 + 4 + \frac{9}{2} - 6 - 2 + 4 = \frac{5}{2}.$$

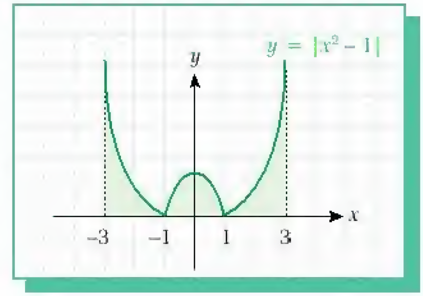
EXAMPLE 21 $\int_{-3}^{3} |x^2-1|\ dx = ?$

Solution If $x^2 - 1 = 0$, then $x = 1$ or $x = -1$.

$$|x^2-1| = \begin{cases} x^2-1, & x \geq 1 \text{ or } x \leq -1 \\ 1-x^2, & -1 < x < 1 \end{cases}$$

$$\int_{-3}^{3} |x^2-1|\, dx = \int_{-3}^{-1} (x^2-1)\ dx + \int_{-1}^{1} (1-x^2)\ dx + \int_{1}^{3} (x^2-1)\ dx$$

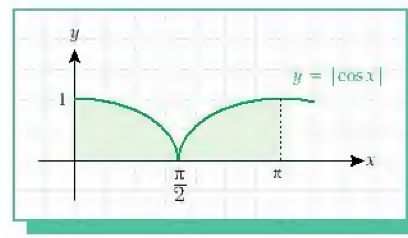
$$= (\frac{x^3}{3} - x)\Big|_{-3}^{-1} + (x - \frac{x^3}{3})\Big|_{-1}^{1} + (\frac{x^3}{3} - x)\Big|_{1}^{3} = \frac{44}{3}.$$



EXAMPLE 22 $\int_{0}^{\pi} |\cos x|\, dx = ?$

Solution

$$|\cos x| = \begin{cases} \cos x & \text{on } [0, \frac{\pi}{2}] \\ -\cos x & \text{on } [\frac{\pi}{2}, \pi] \end{cases}$$



$$\int_{0}^{\pi} |\cos x|\, dx = \int_{0}^{\frac{\pi}{2}} \cos x\ dx + \int_{\frac{\pi}{2}}^{\pi} -\cos x\ dx$$

$$= (\sin x\Big|_{0}^{\frac{\pi}{2}}) + (-\sin x\Big|_{\frac{\pi}{2}}^{\pi})$$

$$= (\sin\frac{\pi}{2} - \sin 0) + (-\sin\pi + \sin\frac{\pi}{2}) = 1 + 1.$$

Check Yourself 4

Evaluate the definite integral of each absolute value function.

a. $\int_{-2}^{2} |x+1|\ dx$ b. $\int_{-2}^{4} |x^2-3x+2|\, dx$ c. $\int_{-\pi}^{\pi} |\sin x|\ dx$

d. $\int_{-1}^{3} \sqrt{x^2-2x+1}\ dx$ e. $\int_{-1}^{3} |x^2-x|\, dx$ f. $\int_{-3}^{4} |x^2-9|\, dx$

Answers

a. 5 b. $\frac{55}{3}$ c. 4 d. 4 e. $\frac{17}{3}$ f. $\frac{118}{3}$

B. INTEGRATING SIGN FUNCTIONS

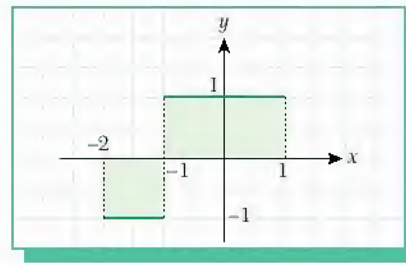
Recall the definition of the sign function:

$$\operatorname{sgn} f(x) = \begin{cases} 1, & f(x) > 0 \\ 0, & f(x) = 0 \\ -1, & f(x) < 0 \end{cases}.$$

When integrating a sign function, we divide the interval into two parts as negative and positive intervals, since $f(x)$ is not continuous at the point where $f(x) = 0$

EXAMPLE 23 $\displaystyle\int_{-2}^{1} \operatorname{sgn}(x+1)\ dx = ?$

Solution $x + 1 = 0 \Rightarrow x = -1$



$$\operatorname{sgn}(x+1) = \begin{cases} 1 & x > -1 \\ 0 & x = -1 \\ -1 & x < -1 \end{cases}$$

$$\int_{-2}^{1} \operatorname{sgn}(x+1)\ dx = \int_{-2}^{-1} -1\, dx + \int_{-1}^{1} 1\, dx$$

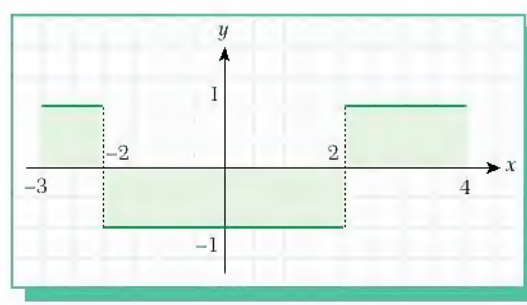
$$= -x \Big|_{-2}^{-1} + x \Big|_{-1}^{1}$$

$$= 1 - 2 + 1 - (-1)$$

$$= 1.$$

EXAMPLE 24 $\displaystyle\int_{-3}^{4} \operatorname{sgn}(x^2-4)\ dx = ?$

Solution $x^2 - 4 = 0 \Rightarrow x = \pm 2$



$$\operatorname{sgn}(x^2-4) = \begin{cases} 1 & x < -2 \text{ or } x > 2 \\ 0 & x = \pm 2 \\ -1 & -2 < x < 2 \end{cases}$$

$$\int_{-3}^{4} \operatorname{sgn}(x^2-4)\ dx = \int_{-3}^{-2} dx + \int_{-2}^{2} -\, dx + \int_{2}^{4} dx$$

$$= x \Big|_{-3}^{-2} + (-x) \Big|_{-2}^{2} + x \Big|_{2}^{4}$$

$$= (-2) - (-3) + (-2)\ -2 + 4 - 2$$

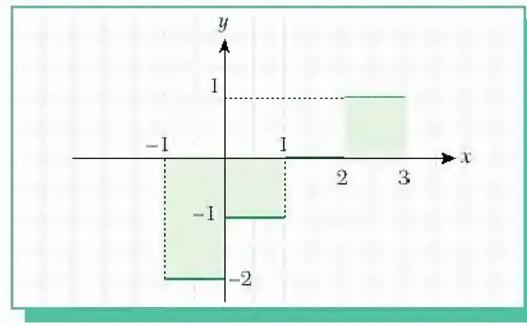
$$= -1.$$

C. INTEGRATING FLOOR FUNCTIONS

The floor function $[\![f(x)]\!]$ is the greatest integer value which is not greater than $f(x)$. If $f(x) \in \mathbb{Z}$ then $[\![f(x)]\!]$ is not continuous. For the points of discontinuity (i.e. for critical points) there is no integral. For this reason, when integrating a floor function we divide the interval into subintervals by using discontinuity points.

EXAMPLE 25 $\int_{-1}^{3} [\![x-1]\!]\,dx = ?$

Solution Look at the graph of $[\![x-1]\!]$.



For $-1 \le x < 0$, $[\![x-1]\!] = -2$.

For $0 \le x < 1$, $[\![x-1]\!] = -1$.

For $1 \le x < 2$, $[\![x-1]\!] = 0$.

For $2 \le x < 3$, $[\![x-1]\!] = 1$.

So $\displaystyle\int_{-1}^{3} [\![x-1]\!]\,dx = \int_{-1}^{0} -2\,dx + \int_{0}^{1} -1\,dx + \int_{1}^{2} 0\,dx + \int_{2}^{3} dx$

$$= -2x\Big|_{-1}^{0} + (-x)\Big|_{0}^{1} + 0 + x\Big|_{2}^{3}$$

$$= 0 - 2 + (-1) - 0 + 3 - 2$$

$$= -2.$$

Check Yourself 5

Evaluate the integrals.

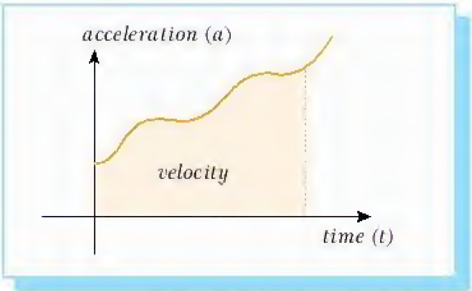
1. a. $\displaystyle\int_{-1}^{2} \operatorname{sgn} x\,dx$ b. $\displaystyle\int_{-3}^{4} \operatorname{sgn}(x-3)\,dx$ c. $\displaystyle\int_{-2}^{3} |\operatorname{sgn}(x^2-1)|\,dx$

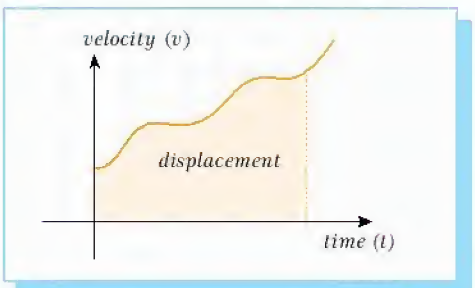
2. a. $\displaystyle\int_{-2}^{2} [\![x-1]\!]\,dx$ b. $\displaystyle\int_{-1}^{3} [\![\frac{x}{2}]\!]\,dx$ c. $\displaystyle\int_{-1}^{2} \operatorname{sgn}\,[\![x]\!]\,dx$

Answers

1. a. 1 b. –5 c. 5 2. a. –6 b. 0 c. 0

Applications in Physics





Integrals have many applications in physics. In fact, some people say that calculus developed because of physics. Newton, one of the fathers of calculus, was also a great physicist.

How can we use integrals in physics? Let us look at some applications.

The area under a velocity-time graph gives us the displacement of a moving object. Also, the area under an acceleration-time graph is the velocity of the object. So we can say that the integral of the function of acceleration is velocity, and the integral of the function of velocity is distance or displacement.

We know that the gravity of acceleration of the earth (g) is about 9.8 m/s^2. Therefore, when you drop an object from a height, its speed after t seconds will be $\int a(t)\,dt$ where $a(t)$ is the acceleration of the earth, g. After integrating this function we get:

$$\text{velocity} = \int a(t)\,dt = \int g\,dt = g \cdot t + c = 9.8 \cdot t + c.$$

How can we calculate the constant c? For this, we use information about the velocity. If we throw an object with a speed v_0 then at time $t = 0$ the velocity will be v_0. Substituting this information in the equation gives us:

$$v = 9.8 \cdot t + c$$

$$v_0 = 9.8 \cdot 0 + c = c.$$

So we have the equation:

$$\text{velocity} = \text{acceleration} \cdot \text{time} + \text{initial velocity}.$$

If we integrate this velocity function with respect to time then we get the distance:

Distance $= \int v(t)\,dt.$

Again by using the given information we can find c from the initial height.

Let us look at two examples of using these formulas.

Example 1 A rocket is launched. Its initial velocity is 120 m/s.

a. What is the velocity of the rocket after five seconds?

b. What is the height of the rocket after five seconds?

c. What is its maximum height?

Solution We can begin by writing $v_0 = 120$ m/s, $g = 9.8$ m/s^2. The rocket moves upward but the acceleration of gravity works downward, so we will take the acceleration of the rocket as negative: $g = -9.8$ m/s^2. Now,

a. $v(t) = \int g\ dt = -g \cdot t + c_1$

$v(0) = -g \cdot 0 + c_1$. This means $c_1 = v_0 = 120$ m/s.

So the velocity at time $t = 5$ is $v(5) = -9.8 \cdot 5 + 120 = 120 - 49 = 71$ m/s.

b. Let $d(t)$ be the distance after time t.

Then $d(t) = \int(-9.8 \cdot t + 120)\ dt = -\frac{1}{2} \cdot 9.8 \cdot t^2 + 120t + c_2$.

We assume that the initial distance of the rocket is 0 m. Therefore when $t = 0$ we have

$d(0) = -\frac{1}{2} \cdot 9.8 \cdot 0^2 + 120 \cdot 0 + c_2 = 0$.

This gives us $c_2 = 0$, which means $d(t) = -\frac{1}{2} \cdot 9.8 \cdot t^2 + 120 \cdot t$.

So we have $d(5) = -\frac{1}{2} \cdot 9.8 \cdot 5^2 + 120 \cdot 5 = 477.5$ m.

c. At the rocket's maximum height its velocity will be 0 m/s. Using the formula from part a we get $v(t) = -9.8 \cdot t + 120 \Rightarrow 0 = -9.8 \cdot t + 120$. This gives $t \cong 12.24$.

So the maximum height $= d(12.24) \cong 734.7$ m.

Example 2 A stone is dropped from a balloon.

a. What is the velocity of the stone after ten seconds?

b. If the stone hits the ground after twenty seconds, what is the height of the balloon?

Solution a. Given information: $g = 9.8$ m/s^2, $v_0 = 0$,

$v(t) = \int g\ dt = g \cdot t + c_1$.

When $t = 0$, $v_0 = 0$ so $c_1 = 0$.

Using $v(t) = g \cdot t$ after ten seconds:

$v(10) = 9.8 \cdot 10 = 98$ m/s.

b. $d(t) = \int 9.8 \cdot t\ dt = \frac{1}{2} \cdot 9.8 \cdot t^2 + c_2$ m. The initial distance is assumed to be 0 so $c_2 = 0$.

So the height of the balloon $= d(20) = \frac{1}{2} \cdot 9.8 \cdot 20^2 = 1960$ m.

EXERCISES 2.2

A. Integrating Absolute Value Functions

1. Integrate the absolute value functions.

a. $\int_{-3}^{2} |x-1| \; dx$

b. $\int_{-5}^{5} |x+3| \; dx$

c. $\int_{-2}^{3} |2x+3| \; dx$

d. $\int_{-2}^{5} |x^2-1| \; dx$

e. $\int_{-3}^{5} |x^2-3x-4|$

f. $\int_{-4}^{5} |x^2+3x+2| \; dx$

g. $\int_{0}^{\pi} |\sin x| \; dx$

h. $\int_{-1}^{2} |x^3| \; dx$

i. $\int_{-4}^{4} \sqrt{x^2+6x+9} \; dx$

B. Integrating Sign Functions

2. Integrate the sign functions.

a. $\int_{-3}^{7} \operatorname{sgn}(x^2-1) \; dx$

b. $\int_{-2}^{2} \operatorname{sgn}(x+1) \; dx$

c. $\int_{-3}^{5} \operatorname{sgn}(x^2-2x+3) \; dx$

d. $\int_{-3}^{4} \operatorname{sgn}(x^3-3x^2-18x+40) \; dx$

e. $\int_{-2}^{3} |\operatorname{sgn}[\![x]\!]| \; dx$

f. $\int_{-1}^{0} \operatorname{sgn}(x^2-5x-6) \; dx$

C. Integrating Floor Functions

3. Integrate the floor functions.

a. $\int_{0}^{2} [\![x+1]\!] \; dx$

b. $\int_{-1}^{2} [\![2x-1]\!] \; dx$

c. $\int_{1}^{2} x \cdot [\![x+2]\!] \; dx$

d. $\int_{0}^{3} [\![\frac{x}{2}]\!] \; dx$

e. $\int_{-1}^{3} [\![3x]\!] \; dx$

f. $\int_{-2}^{2} |x| \cdot x^{[\![\frac{x}{2}]\!]} \; dx$

g. $\int_{-2}^{2} |\operatorname{sgn}[\![1-x]\!]| \; dx$

h. $\int_{2}^{3} x \cdot [\![2x+5]\!] \; dx$

CHAPTER SUMMARY

- Definition of the Definite Integral

a. Let $f(x)$ be a continuous function defined on the interval $[a, b]$. Then the area between the graph of $f(x)$ and x-axis is called the definite integral of $f(x)$.

b. If the graph is under the x-axis then its integral will be negative. However, the area A is a positive value:

$$A = -\int_a^b f(x)\, dx.$$

- The Fundamental Theorem of Calculus

If $f(x)$ is a function such that $f: [a, b] \to R$.

If $F'(x) = f(x)$ and $\int f(x)\, dx = F(x) + c$, then

$$\int_a^b f(x)\, dx = F(b) - F(a).$$

- Properties of the Definite Integral

Let $f: [a, b] \to R$ and $g: [a, b] \to R$ be two integrable functions. Then the following properties hold:

1. $\int_a^a f(x)\, dx = 0$

2. $\int_a^b f(x)\, dx = -\int_b^a f(x)\, dx$

3. $\int_a^b c \cdot f(x)\, dx = c \cdot \int_a^b f(x)\, dx$

4. $\int_a^b [f(x) \pm g(x)]\, dx = \int_a^b f(x)\, dx \pm \int_a^b g(x)\, dx$

5. If $a \le b \le c$ then $\int_a^c f(x)\, dx = \int_a^b f(x)\, dx + \int_b^c f(x)\, dx$.

6. If $f(x) \ge g(x)$, then $\int_a^b f(x)\, dx \ge \int_a^b g(x)\, dx$.

7. $\left| \int_a^b f(x)\, dx \right| \le \int_a^b |f(x)|\, dx$

- Leibniz's Rule

If $f: [a, b] \to R$ is a continuous function and $F(x) = \int_a^x f(t)\, dt$ then $F'(x) = \dfrac{d}{dx}\left(\int_a^x f(t)\, dt\right) = f(x)$.

This implies that for any two differentiable functions $u(x)$ and $v(x)$:

a. $F(x) = \int_a^{v(x)} f(t)\, dt \Rightarrow F'(x) = f(v(x)) \cdot v'(x)$.

b. $F(x) = \int_{u(x)}^{v(x)} f(t)\, dt$

$\Rightarrow F'(x) = f(v(x)) \cdot v'(x) - f(u(x)) \cdot u'(x)$.

- The Mean Value Theorem

Let $f: [a, b] \to R$ be a continuous function. Then there exists at least one $c \in [a, b]$ such that

$$f(c) = \frac{\int_a^b f(x)\, dx}{b - a}.$$

The real number c is called the mean value of $f(x)$ on the interval $[a, b]$.

- Integrating Absolute Value Functions

$$|f(x)| = \begin{cases} f(x), & f(x) \ge 0 \\ -f(x), & f(x) < 0 \end{cases}$$

To integrate an absolute value function, first find the positive and negative parts of the given function. For the intervals where $f(x)$ changes sign, divide the interval into two or more subintervals by using the fifth property of definite integrals and change the negative parts to positive.

- Integrating Sign Functions

$$\operatorname{sgn} f(x) = \begin{cases} 1, & f(x) > 0 \\ 0, & f(x) = 0 \\ -1, & f(x) < 0 \end{cases}$$

To integrate a sign function, divide the interval into two parts as negative and positive intervals, since $f(x)$ is not continuous at the point where $f(x) = 0$.

- Integrating Floor Functions

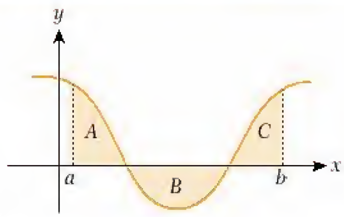
$[\![f(x)]\!]$ is the greatest integer value which is not greater than $f(x)$. If $f(x) \in Z$ then $[\![f(x)]\!]$ is not continuous. For the points of discontinuity (i.e. for critical points) there is no integral, so divide the interval into subintervals by using the discontinuity points.

Concept Check

- What is a definite integral?
- What is the relation between the definite integral of a function and the area between the x-axis and the graph of the function?
- What is the Fundamental Theorem of Calculus?
- How can we explain Leibniz's Rule?
- How can we find the mean value of a function on a closed interval $[\boldsymbol{a}, \boldsymbol{b}]$?
- Name three special types of function and the integration methods we use for them.
- Why do we need to divide the interval into parts when we integrate these special functions?

1. In the figure, A, B and C are areas such that $A = 3$ unit^2, $B = 6$ unit^2, and $C = 5$ unit^2.

What is $\int_a^b f(x)\,dx$?



A) 27 B) 14 C) 4 D) 2 E) 8

2. $\int_1^3 x^2 \, dx = ?$

A) $\frac{26}{3}$ B) $\frac{7}{9}$ C) 5 D) 8 E) 0

3. $\int_0^1 5 \cdot e^{2x-4} \, dx = ?$

A) $\frac{5e^2}{4}$ B) $\frac{5e^{-2}}{2}$ C) $\frac{5e^2-5}{2e^4}$ D) $\frac{e^2-1}{10e^4}$ E) $\frac{e^{-4}-e^{-2}}{2}$

4. $\int_0^{\pi/2} \sin x \, dx = ?$

A) 0 B) 1 C) –1 D) $\frac{1}{2}$ E) $\frac{\sqrt{3}}{2}$

5. $\int_7^7 3x^7 \cdot \cos x \cdot \ln x \, dx = ?$

A) –1 B) 0 C) 1 D) $\cos(\ln x)$ E) $\frac{\ln 7}{\sin 7}$

6. $\frac{d}{dx}\left(\int_{-2}^{7} (4x^2+3x-4)\, dx\right) = ?$

A) –5 B) 3 C) $3x-4$ D) $\frac{1}{x-2}$ E) 0

7. $\int_1^e \frac{x^2-2x+3}{x} \, dx = ?$

A) 1 B) 0 C) $\frac{\ln 4 - e^2}{2}$ D) $\frac{e^2-4e+9}{2}$ E) $\frac{e^2+e-1}{2}$

8. $\int_0^3 x^3 \, dx = ?$

A) 9 B) $\frac{9}{2}$ C) $\frac{81}{4}$ D) $\frac{17}{3}$ E) 0

9. $\int_0^1 (2x+1)(x^2+x-4)\ dx = ?$

A) 6 B) –4 C) 2 D) 4 E) –6

10. $\int_0^{\pi/6} \sin 2x\ dx = ?$

A) 1 B) 0 C) $-\frac{1}{2}$ D) $\frac{\sqrt{3}}{2}$ E) $\frac{1}{4}$

11. $\int_1^2 (3x-1)^2\ dx = ?$

A) 13 B) $\frac{7}{6}$ C) $\frac{17}{3}$ D) $\frac{31}{3}$ E) $\frac{321}{5}$

12. $\int_0^1 x \cdot e^x\ dx = ?$

A) 0 B) –1 C) e^2 D) 1 E) $-e$

13. $\int_1^3 \frac{1}{(2x-1)^3}\ dx = ?$

A) $\frac{2}{13}$ B) $\frac{6}{25}$ C) $\frac{1}{7}$ D) $\frac{1}{25}$ E) $\frac{7}{125}$

14. $\int_0^{\pi/3} \frac{1-\cos 2x}{\sin 2x}\ dx = ?$

A) 1 B) 0 C) $\ln 2$ D) $\frac{1}{2}$ E) $\frac{\sqrt{3}}{2}$

15. $F(x) = \int_3^{x^2} \cos(3t)\ dt$ is given. What is $F'(0)$?

A) 0 B) 1 C) $\frac{1}{\sqrt{3}}$ D) $\frac{\sqrt{2}}{2}$ E) $\frac{1}{4}$

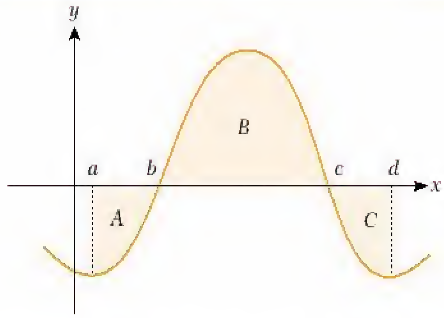
16. What is the mean value of the function $f(x) = \cos x$ on the interval $\left[\frac{\pi}{4}, \frac{\pi}{2}\right]$?

A) 0 B) $\frac{\sqrt{2}}{2\pi}$ C) $\frac{1-3\sqrt{2}}{\pi}$

D) $\frac{4-2\sqrt{2}}{\pi}$ E) $\frac{\pi}{4}(1-\sqrt{2})$

CHAPTER REVIEW TEST 2B

1.



In the figure, $\int_b^c f(x)\ dx = 18$ and $\int_a^d f(x)\ dx = 10$.

What is the total area of the shaded region?

A) 28 B) 8 C) 10 D) 12 E) 26

2. $\int_0^2 (3x^2 - 4x)\ dx = ?$

A) 0 B) 4 C) 12 D) 16 E) –4

3. $\int_0^1 e^{4x}\ dx = ?$

A) 0 B) $\frac{1}{4}$ C) $\frac{e}{4}$ D) $e^4 - 1$ E) $\frac{e^4 - 1}{4}$

4. $\int_{\pi/4}^{\pi/2} \cos x \cdot \sin x\ dx = ?$

A) $\frac{\sqrt{2}}{2}$ B) $\frac{1}{2}$ C) 1 D) $\frac{1}{4}$ E) 0

5. $\int_a^b (x+2)\ dx = 10$ and $a - b = 2$. What is a?

A) 2 B) –6 C) –4 D) 5 E) 6

6. $\int_0^\pi (\sin x + \cos x)\ dx = ?$

A) 0 B) 1 C) 2 D) 4 E) –1

7. $\int_0^7 \sqrt{9+x}\ dx = ?$

A) $\frac{74}{3}$ B) $\frac{16}{3}$ C) $\frac{7}{3}$ D) $\frac{7}{4}$ E) $\frac{128}{5}$

8. $\int_2^{e+1} \frac{x}{x-1}\ dx = ?$

A) $e + 3$ B) $\frac{e^2 - 1}{4}$ C) $\frac{1+e}{2}$

D) e E) $1 - e$

9. $\int_{1}^{e^2} \ln x \; dx = ?$

A) $\dfrac{1+e^2}{3}$ B) $\dfrac{2e^2-1}{2}$ C) $e^2 + 1$

D) 23 E) $1 - e$

10. $\int_{0}^{1} \dfrac{d(x^3)}{1+x^3} = ?$

A) 0 B) $\ln 2$ C) 1

D) $\ln 3 - \ln 2$ E) e^2

11. $\int_{1}^{3} \dfrac{2x+3}{2x-1} \; dx = ?$

A) $\ln 9 - \ln 3$ B) $\ln 3 - \ln 5$ C) 0

D) $2 + \ln 25$ E) $\ln 5 - 1$

12. $\int_{1}^{2} \dfrac{2x-3}{x^2+3x} \; dx = ?$

A) $2\ln 3 + \ln 5$ B) $\ln(\dfrac{7}{4})$ C) $\ln(\dfrac{125}{12})$

D) $1 + \ln 2$ E) $\ln(\dfrac{125}{128})$

13. $\int_{0}^{\pi/3} \sin^3 x \cdot \cos^3 x \; dx = ?$

A) $\dfrac{17}{43}$ B) $\dfrac{9}{128}$ C) $\dfrac{58}{161}$ D) $\dfrac{1}{2}$ E) -1

14. $F(x) = \int_{1}^{x} \dfrac{t^2}{1+t^3} \, dt$ is given. What is $F'(x)$?

A) $\dfrac{x^2}{1+x^3}$ B) $\dfrac{2x^2}{1+x^3}$ C) $\dfrac{2x}{1+3x^2}$

D) 0 E) $\ln|1 + x|$

15. $\int_{0}^{\pi/4} \cos x \sin^3 x \, dx = ?$

A) 0 B) 1 C) $\dfrac{1}{16}$ D) $\dfrac{1}{4}$ E) $\dfrac{1}{2}$

16. What is the mean value of the function $f(x) = 2x^2 + 1$ on the interval $[0, 5]$?

A) $\dfrac{27}{4}$ B) $\dfrac{48}{5}$ C) $\dfrac{53}{3}$ D) $\dfrac{62}{7}$ E) $\dfrac{17}{5}$

Chapter 3

APPLICATIONS OF DEFINITE INTEGRALS

1 FINDING THE AREA UNDER A CURVE

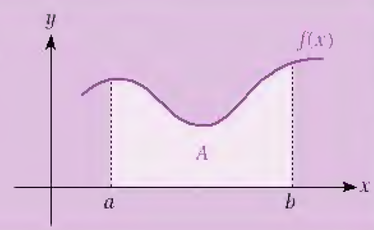
Definite integrals have many applications. We can use definite integrals to find the area of a region bounded by a curve, the volume created by rotating a curve, the length of a curve, the mean value of a function, the center of mass of an object, and to calculate the displacement and work done in motion and projectile problems.

Most of these applications are useful in mathematics or physics. In this chapter we will look at the applications of the definite integral in mathematics.

FINDING AREA

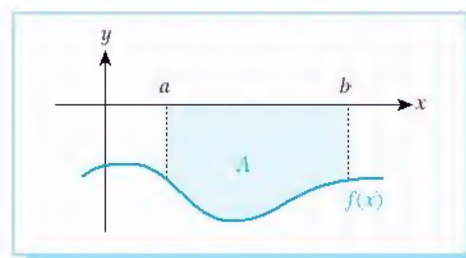
Let $f: [a, b] \to R$ be a continuous function such that for every x, $f(x) \geq 0$. Then the area of the region between $y = f(x)$ and the x-axis on the interval $[a, b]$ is

$$A = \int_a^b f(x)\ dx.$$



Conclusions

The theorem above gives us four important results.



1. If $f(x) \leq 0$ then the area of the region between $y = f(x)$ and the x-axis on the interval $[a, b]$ will be

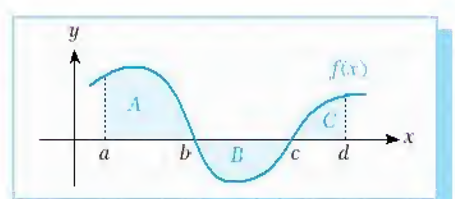
   $$A = \int_a^b -f(x)\ dx = -\int_a^b f(x)\ dx.$$

   More generally, if $y = f(x)$ is any continuous function on the interval $[a, b]$ then the area between $y = f(x)$ and the x-axis is

   $$A = \int_a^b |\ f(x)\ |\ \ dx.$$

2. In the figure, the total area of the region will be

   $$\text{Area} = A + B + C = \int_a^b f(x)\ dx - \int_b^c f(x)\ dx + \int_c^d f(x)\ dx.$$



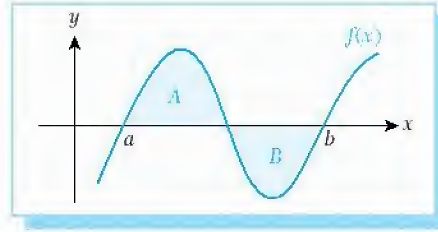
Remember:

a. The definite integral of $f(x)$ on $[a, b]$ is

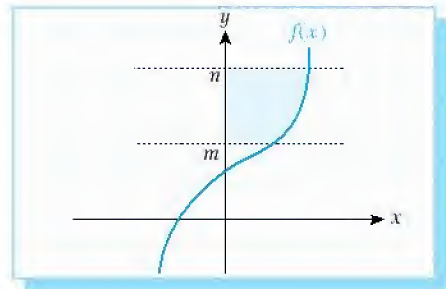
$$\int_a^b f(x)\,dx = A - B.$$

b. The area between $f(x)$ and the x-axis on $[a, b]$ is

Area $= A + B$.

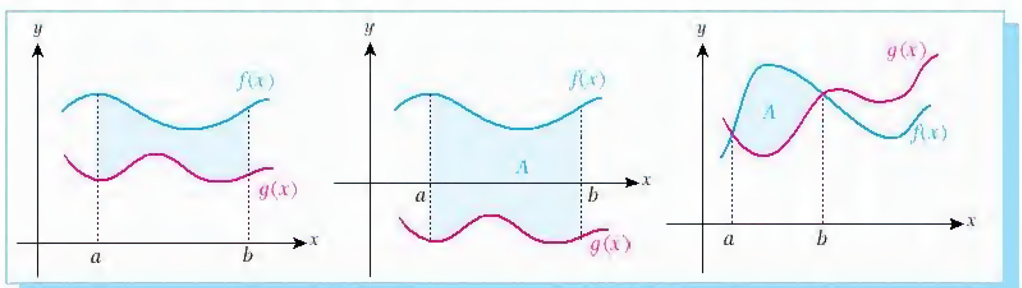


3. The area between $y = f(x)$, the y-axis and the lines $y = m$ and $y = n$ is $A = \int_m^n f(y)\,dy$.



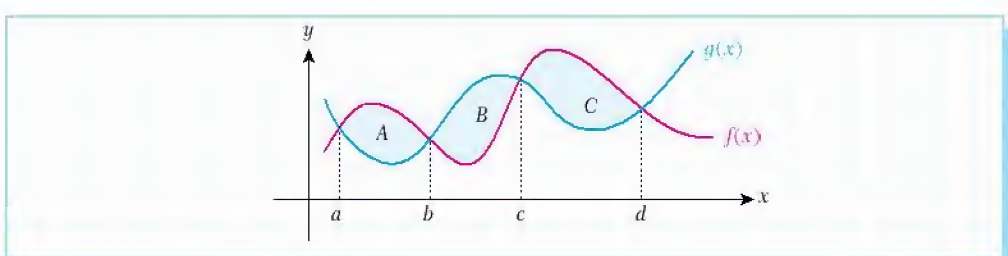
4. We can use the formula for the area under a curve to define the area between two curves.

a.



Let $f(x)$ and $g(x)$ be two curves. Then the area A between $f(x)$ and $g(x)$ on the interval $[a, b]$ is $A = \int_a^b [f(x) - g(x)]\,dx$.

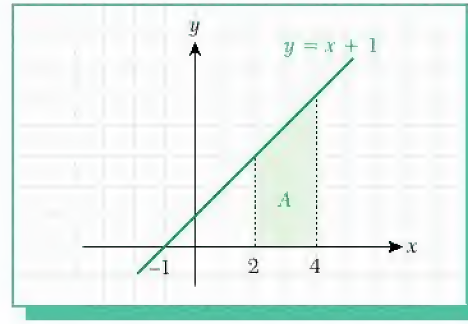
b.



The area between $f(x)$ and $g(x)$ is

$$A + B + C = \int_a^b (f(x) - g(x))\,dx + \int_b^c (g(x) - f(x))\,dx + \int_c^d (f(x) - g(x))\,dx.$$

EXAMPLE 1 Find the area A of the region bounded by the graph of $y = x + 1$, the x-axis, and the lines $x = 2$ and $x = 4$.

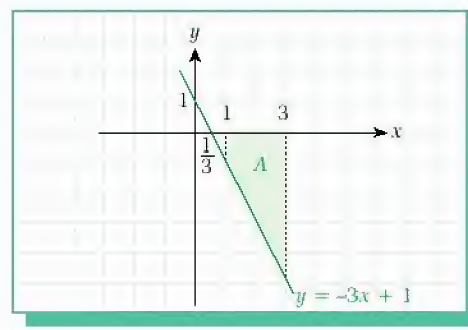


Solution $A = \int_2^4 (x+1)\,dx = \left(\frac{x^2}{2} + x\right)\Big|_2^4$

$= (8+4) - (2+2)$

$= 8.$

EXAMPLE 2 Find the area A bounded by the graph of $y = -3x + 1$, the x-axis, and the lines $x = 1$ and $x = 3$.



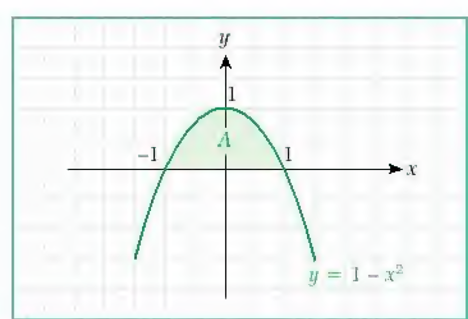
Solution $A = \int_1^3 -(-3x+1)\,dx = \left(\frac{3x^2}{2} - x\right)\Big|_1^3$

$= (\frac{27}{2} - 3) - (\frac{3}{2} - 1)$

$= \frac{21}{2} - \frac{1}{2}$

$= 10.$

EXAMPLE 3 What is the area of the finite region between the graph of $y = 1 - x^2$ and the x-axis?



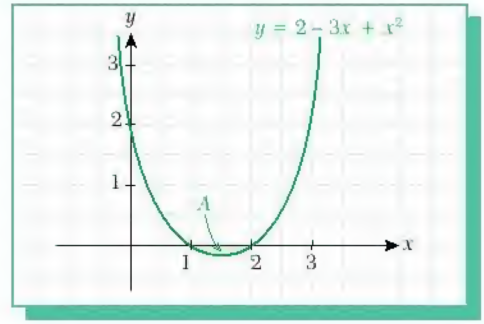
Solution $f(x) = 1 - x^2 = 0 \Rightarrow x = \pm 1$

$A = \int_{-1}^{1} (1 - x^2)\,dx = \left(x - \frac{x^3}{3}\right)\Big|_{-1}^{1}$

$= (1 - \frac{1}{3}) - (-1 + \frac{1}{3})$

$= \frac{4}{3}.$

EXAMPLE 4 What is the area of the finite region between the graph of $y = 2 - 3x + x^2$ and the x-axis?



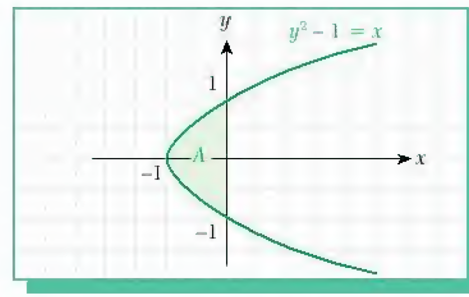
Solution $2 - 3x + x^2 = 0 \Rightarrow x_1 = 1,\ x_2 = 2$

$$A = \int_1^2 -(2 - 3x + x^2)\,dx = \left(-2x + \frac{3x^2}{2} - \frac{x^3}{3}\right)\Bigg|_1^2$$

$$= (-4 + 6 - \frac{8}{3}) - (-2 + \frac{3}{2} - \frac{1}{3})$$

$$= \frac{1}{6}.$$

EXAMPLE 5 Find the area of the region bounded by the graph of $y^2 - 1 = x$ and the y-axis.



Solution $y^2 - 1 = 0 \Rightarrow y = \pm 1$

$$A = \int_{-1}^{1} -(y^2 - 1)\,dy = -(\frac{y^3}{3} - y)\Bigg|_{-1}^{1}$$

$$= -(\frac{1}{3} - 1) + (-\frac{1}{3} + 1)$$

$$= \frac{4}{3}.$$

Check Yourself 1

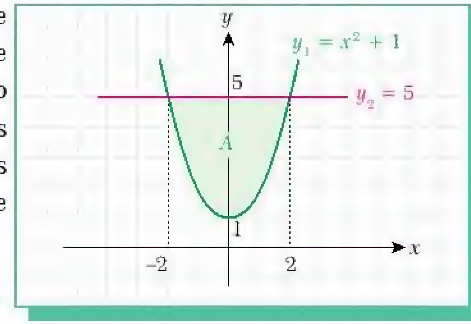
1. Find the finite area between the graph of $y = x^2 - 4$ and the x-axis.
2. Find the area of the region bounded by $y = 2x^2 - 4x + 5$, the x-axis, and the lines $x = 2$ and $x = 3$.
3. What is the area of the region bounded by $y = 3x + 5$, the x-axis, and the lines $x = 1$ and $x = 4$?
4. Find the area of the region bounded by the graphs of $y = 5x + 1$, the y-axis, and the lines $y = 2$ and $y = 3$.
5. Find the area of the region bounded by the graph of $y^2 = x + 9$ and the y-axis.
6. What is the area of the region bounded by $y = 2^x$ and the lines $x = 1$, $x = -1$ and $y = 0$?

Answers

1. $\frac{32}{3}$ 2. $\frac{23}{3}$ 3. $\frac{75}{2}$ 4. $\frac{3}{10}$ 5. 36 6. $\frac{3}{2\ln 2}$

EXAMPLE 6 Find the area of the finite region between the graphs of $y_1 = x^2 + 1$ and $y_2 = 5$.

Solution Before we begin, we need to find the interval for the integration. If we sketch a rough graph we can see that the interval is set by the intersection of the two lines. Therefore, we need to solve the two functions simultaneously to find the upper and lower limits for the integral. Then we can find the definite integral between the limits.



$$\left.\begin{array}{l} y_1 = x^2 + 1 \\ y_2 = 5 \end{array}\right\} \; y_1 = y_2 \Rightarrow x^2 + 1 = 5 \Rightarrow x = \pm 2$$

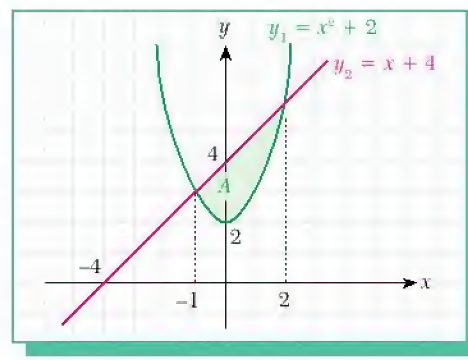
$$A = \int_{-2}^{2} (5 - (x^2 + 1))\, dx = \int_{-2}^{2} (4 - x^2)\, dx = \left(4x - \frac{x^3}{3}\right)\Big|_{-2}^{2} = (8 - \frac{8}{3}) - (-8 + \frac{8}{3}) = \frac{32}{3}.$$

EXAMPLE 7 Find the area of the region bounded by $y_1 = x^2 + 2$ and $y_2 = x + 4$.

Solution Again, we find the interval by solving the two functions simultaneously.

$y_1 = y_2 \Rightarrow x^2 + 2 = x + 4$

$x_1 = 2$ and $x_2 = -1$



$$A = \int_{-1}^{2} ((x+4) - (x^2+2))\, dx = \int_{-1}^{2} (-x^2 + x + 2)\, dx$$

$$= \left(-\frac{x^3}{3} + \frac{x^2}{2} + 2x\right)\Big|_{-1}^{2} = (-\frac{8}{3} + 2 + 4) - (\frac{1}{3} + \frac{1}{2} - 2)$$

$$= \frac{9}{2}.$$

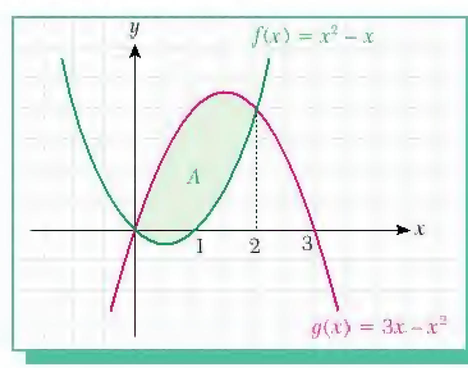
EXAMPLE 8 Find the area of the region bounded by $f(x) = x^2 - x$ and $g(x) = 3x - x^2$.

Solution

$f(x) = g(x)$

$x^2 - x = 3x - x^2$

So $x_1 = 0,\ x_2 = 2.$

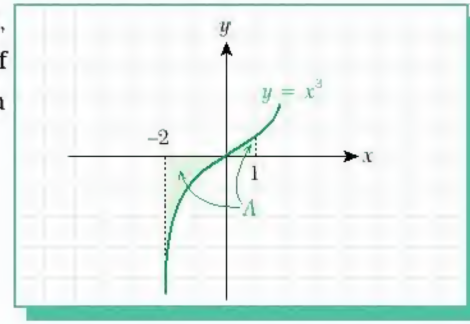


$$A = \int_{0}^{2} ((3x - x^2) - (x^2 - x))\, dx = \int_{0}^{2} (-2x^2 + 4x)\, dx$$

$$= \left(-\frac{2x^3}{3} + 2x^2\right)\Big|_{0}^{2} = -\frac{16}{3} + 8 - 0 = \frac{8}{3}.$$

EXAMPLE 9 Find the finite area between the graphs of $y = x^3$, $y = 0$, $x = -2$ and $x = 1$.

Solution The graph intersects the x-axis at the point $x = 0$, so part of the area lies above the x-axis and part of it is below the x-axis. Therefore we divide the area into two parts.

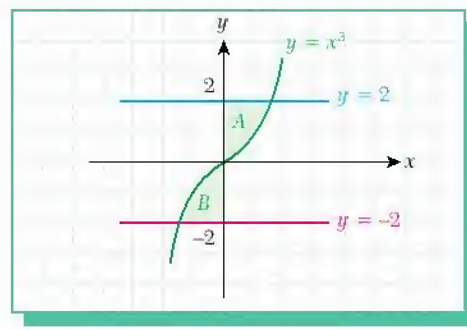


$$A = \int_{-2}^{0} -x^3\,dx + \int_{0}^{1} x^3\,dx = -\frac{x^4}{4}\Big|_{-2}^{0} + \frac{x^4}{4}\Big|_{0}^{1}$$

$$= 0 - (-4) + \frac{1}{4} - 0 = \frac{17}{4}.$$

EXAMPLE 10 Find the area of the region bounded by $y = x^3$, $x = 0$, $y = -2$ and $y = 2$.

Solution $y = x^3 \Rightarrow x = \sqrt[3]{y}$.



$$A = \int_{0}^{2} \sqrt[3]{y}\,dy = \int_{0}^{2} y^{\frac{1}{3}}\,dy = \frac{3y^{\frac{4}{3}}}{4}\Big|_{0}^{2} = \frac{3\sqrt[3]{2}}{2}$$

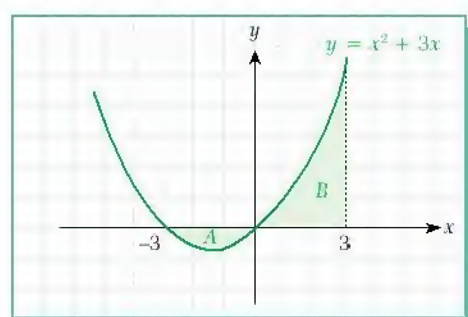
$$B = \int_{-2}^{0} \sqrt[3]{y}\,dy = -\frac{3y^{\frac{4}{3}}}{4}\Big|_{-2}^{0} = \frac{3\sqrt[3]{2}}{2}$$

Total area $= A + B = \dfrac{3\sqrt[3]{2}}{2} + \dfrac{3\sqrt[3]{2}}{2} = 3\sqrt[3]{2}$.

These two areas are symmetric. Therefore, if we multiply A by 2 we will find the total area.

EXAMPLE 11 What is the area of the region bounded by $y = x^2 + 3x$, the x-axis and the line $x = 3$?

Solution



$$A + B = \int_{-3}^{0} -(x^2 + 3x)\,dx + \int_{0}^{3} (x^2 + 3x)\,dx$$

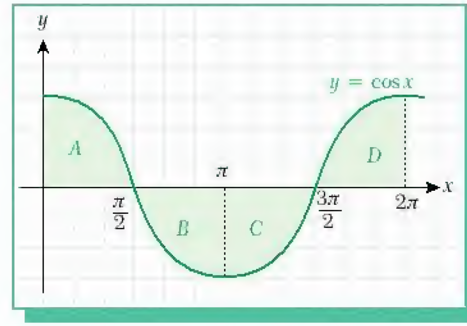
$$= \left(-\frac{x^3}{3} - \frac{3x^2}{2}\right)\Big|_{-3}^{0} + \left(\frac{x^3}{3} + \frac{3x^2}{2}\right)\Big|_{0}^{3}$$

$$= 0 - (9 - \frac{27}{2}) + \left(9 + \frac{27}{2}\right) - 0$$

$$= 27.$$

EXAMPLE 12 What is the area of the region between $y = \cos x$ and the x-axis on the interval $[0, 2\pi]$?

Solution We can see from the figure that the areas of all four parts A, B, C, and D are equal. So we can calculate the answer as follows:



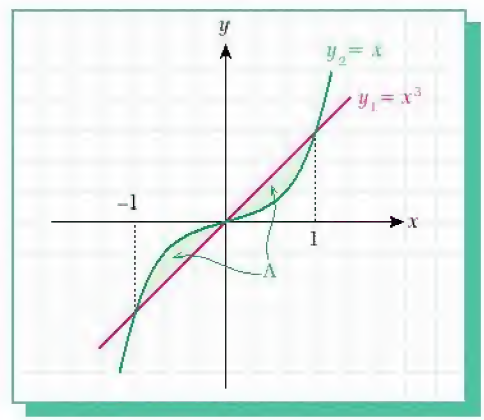
$$\text{Total area} = 4 \cdot \int_0^{\pi/2} \cos x\, dx = 4\sin x \Big|_0^{\pi/2}$$

$$= 4\sin\frac{\pi}{2} - 4\sin 0$$

$$= 4.$$

EXAMPLE 13 What is the area of the region bounded by the graphs of the functions $y_1 = x^3$ and $y_2 = x$?

Solution First we need to find the intersection points:



$y_1 = y_2$

$x^3 = x$; $x^3 - x = 0$; $x\,(x - 1)(x + 1) = 0$.

So the intersection points are $x = -1, 0, 1$.

$$A = \int_{-1}^{0} (x^3 - x)\, dx + \int_0^1 (x - x^3)\, dx$$

$$= \left(\frac{x^4}{4} - \frac{x^2}{2}\right)\Big|_{-1}^{0} + \left(\frac{x^2}{2} - \frac{x^4}{4}\right)\Big|_0^1 = \frac{1}{2}.$$

Check Yourself 2

1. Find the area of the region bounded by $y = x^2 + 1$ and $y = 3x - 1$.
2. What is the area of the region bounded by $y = 3x^2 + 1$ and $y = 4$?
3. Find the area of the region bounded by $y = 2x^2 + 3x$ and $y = -x^2 - 3x + 24$.
4. What is the area of the region bounded by the graphs of $y = x^3 - 2$ and $y = -2x^2 + x$?

Answers

1. $\frac{1}{6}$ 2. 4 3. 108 4. $\frac{37}{12}$

EXERCISES 3.1

1. Find the area of the region bounded by the graph of $y = 4 - x^2$ and the x-axis.

2. Find the area of the region bounded by the line $y = 3 + x$ and the x and y axes.

3. Find the area of the region bounded by $y = 2x - 1$ and the x-axis on the interval $[0, 3]$.

4. Find the area of the region bounded by the graph of $y = 1 - 3x$ and the x-axis on the interval $[2, 5]$.

5. Find the area of the region bounded by the graph of $y = x^2 + 5$ and the x-axis on the interval $[0, 3]$.

6. Find the area of the region bounded by the graph of $y = x^2 - 3x - 4$ and the x-axis on the interval $[-1, 7]$.

7. Find the area of the region bounded by the graphs of $y = x - 1$, $y = 0$, $x = 1$ and $x = 3$.

8. Find the area of the region bounded by the graphs of $y = x^3 - 1$, $y = 0$, $x = 1$ and $x = 2$.

9. Find the area of the region bounded by the graphs of $y = 3 - x^2$, $y = 0$, $x = 0$ and $x = 2$.

10. Find the area of the region bounded by the graphs of $y = 2x + 1$, $x = 0$, $y = 1$ and $y = 3$.

11. Find the area of the region bounded by the graph of $y = 3x - 1$, the y-axis, and the lines $y = 0$ and $y = 2$.

12. Find the area of the region bounded by the graph of $y = \sqrt{x}$, the y-axis, and the lines $y = 1$ and $y = 3$.

13. Find the area bounded by the graph of $x = y^2 - 4$ and the y-axis.

14. Find the area of the region bounded by the graph of $x = y^2 - 3y + 2$ and the y-axis.

15. Find the area of the region bounded by the graphs of $y = 2x^2 - 3x + 1$ and $y = 3$ on the interval $[2, 3]$.

16. Find the area of the region bounded by the graphs of $y = 2x - 5$, $y = -2$, $x = 1$ and $x = 3$.

17. Find the area of the region bounded by the curves $y = x^2 - 1$ and $y = 1 - x^2$.

18. Find the area of the region bounded by the curves $f(x) = x^3$ and $g(x) = \sqrt{x}$.

19. Find the area of the region bounded by the curves $y = 4 - x^2$ and $y = x^2 + 2$.

20. Find the area of the region bounded by the graphs of $y = 2x^2 - 3x + 1$ and $y = -8x + 4$.

21. Find the area of the region bounded by the graphs of $y = x^2 - 1$ and $y = 3x + 3$.

22. Find the area of the region bounded by the graphs of $y = 4 - x^2$ and $y = 2x + 1$.

23. Find the area of the region bounded by the graphs of $y = x^3 - 2x^2$ and $y = 3x$.

24. Find the area of the region bounded by the graphs of $f(x) = x^2$ and $g(x) = 4$.

25. Find the area of the region bounded by the graphs of $f(x) = 3 - x^2$ and $g(x) = 2$.

26. Find the area of the region bounded by the graphs of $y = x^2$ and $x = y^2$.

27. Find the area of the region bounded by the graphs of $y = \cos x$ and $y = \sin x$ on the interval $[0, \pi]$

28. Find the area of the region bounded by the graphs of $y = \sin x$ and $y = \cos 2x$ on $[0, \frac{\pi}{3}]$.

29. Find the area of the region bounded by the graphs of $y = \frac{\cos x}{3}$ and $y = \frac{\sin x}{3}$ on $[\frac{\pi}{3}, \frac{\pi}{2}]$.

30. Find the area of the region bounded by the graphs of $y = 2\sin x$ and $y = 3\cos x$ on $[0, \frac{\pi}{6}]$.

31. Find the area of the region bounded by the curve $y = \sin x$ and the x-axis on $[0, 2\pi]$.

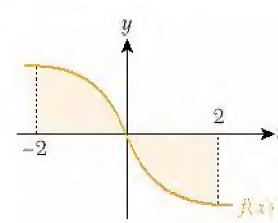
32. Find the area of the region between the graph of $y = 5 \cdot \cos 4x$ and the x-axis on $[0, \frac{\pi}{4}]$.

33. Find the area of the region bounded the graph of $y = 3 \cdot \sin x$, the x-axis, and the lines $x = 0$ and $x = \frac{\pi}{3}$.

34. What is the area of the region bounded by the graphs of $y = x^2$, $y = 3x^2$ and $y = 4x$?

35. In the figure the shaded area is 12 cm^2 and $\int_{-2}^{2} f(x)\,dx = 0$.

What is $\int_{0}^{2} f(x)\,dx$?

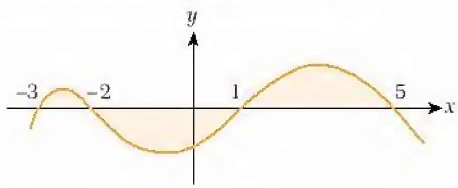


36. The area of the region bounded by $y = ax^2$ $(a > 0)$, the x-axis and the line $x = 3$ is 18 cm^2. What is the value of a?

37. Find the area of the region bounded by the graph of $y = \frac{1}{x}$, the x-axis, and the graphs $x = 1$ and $x = e^3$.

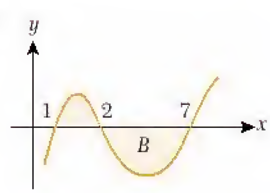
38. Find the area of the region bounded by the graph of $y = \ln x$, the x-axis, and the graphs $x = e$ and $x = e^2$.

39. The figure shows the graph of the function $f(x)$. $\int_{-2}^{1} f(x)\,dx = -5$ and $\int_{-3}^{5} f(x)\,dx = 5$ are given. Find the total area of the shaded region.



40. The figure shows the graph of $f(x)$.

If $\int_{1}^{7} f(x)\,dx = -3$ and the area B is 8 cm^2, what is $\int_{1}^{7} |f(x)|\,dx$?



41. Find the area of the region bounded by the graphs of $y = x^2 + 3x - 1$, $y = 0$, $x = -3$ and $x = 0$.

42. Find the area of the region bounded by the graphs of $y = x^2$ and $y = 1$ on the interval $[-2, 1]$.

43. Find the area of the region bounded by the graphs of $y = x^3 + 1$ and $y = 5$ on the interval $[0, 2]$.

44. Find the area of the region bounded by the graphs of $y = 1 + 3x$, $y = 8$, $x = 2$ and $x = 3$.

45. Find the area of the region bounded by the curves $y = 2x^2 - 3x + 5$ and $y = 10 - x - x^2$.

46. Find the area of the region bounded by the curves $y = x^3 + x^2 + 2x$ and $y = 7x^2 - 9x + 6$.

47. Find the area of the region bounded by the graphs of $x = y^2$ and $y = x - 3$.

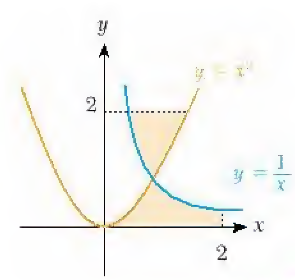
48. Find the area of the region bounded by the graphs of $y = \sin 3x$, $y = 2\cos x$, $x = 0$ and $x = \frac{\pi}{2}$.

49. Find the area of the region bounded by the graphs of $y = \cos 2x$ and $y = 2\sin x$ on $\left[\frac{5\pi}{3}, \frac{7\pi}{4}\right]$.

50. What is the area of the region bounded by the tangent line of $f(x) = \ln x$ at the point $x = e$, the graph of the function $g(x) = \ln x$, and the x-axis?

51. Find the area of the region bounded by $y^2 = x$, $y = \frac{1}{8}x^2$, $y = 1$, $y = \frac{3}{2}$.

52. Find the area of the shaded region in the figure.



2 OTHER APPLICATIONS

A. CALCULATING THE VOLUME OF A SOLID OF REVOLUTION

When a region is rotated about an axis we obtain a solid figure. This figure is called a solid of revolution. We can use the definite integral to find the volume of a solid of revolution.

FINDING VOLUME

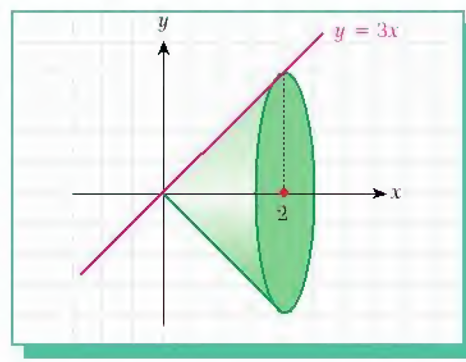
Let $f(x)$ be a continuous function defined on $[a, b]$. Then the volume V of the solid generated by rotating the area between the graph of $f(x)$ and the x-axis on $[a, b]$ about the x-axis is

$$V = \pi \cdot \int_a^b [f(x)]^2 \, dx.$$

Note

f^2x is used in the place of $[f(x)]^2$.

EXAMPLE 14 What is the volume V of the solid figure generated by rotating the area between $y = 3x$ and the x-axis around the x-axis on the interval $[0, 2]$?



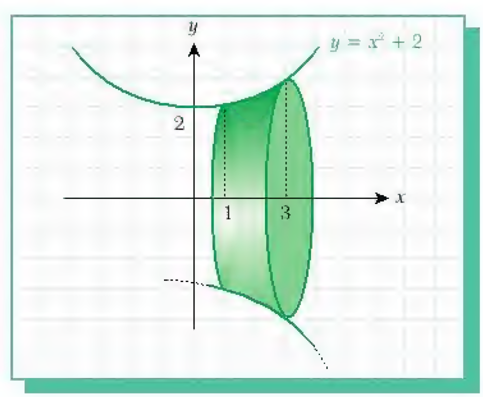
Solution Look at the figure. Using the theorem we can write

$$V = \pi \cdot \int_0^2 (3x)^2 \, dx = \pi \cdot \int_0^2 9x^2 \, dx$$

$$= 3\pi x^3 \Big|_0^2$$

$$= 24\pi.$$

EXAMPLE 15 What is the volume V of the solid obtained by rotating the region between $y = x^2 + 2$ and the x-axis around the x-axis on the interval $[1, 3]$?



Solution $V = \pi \cdot \int_1^3 (x^2 + 2)^2 \, dx$

$$= \pi \cdot \int_1^3 (x^4 + 4x^2 + 4) \, dx$$

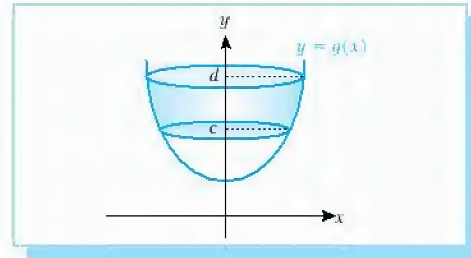
$$= \pi \cdot \left(\frac{x^5}{5} + \frac{4x^3}{3} + 4x\right) \Big|_1^3$$

$$= \frac{1366\pi}{15}$$

Note

If we rotate a figure around the y-axis then the volume is created by $x = f(y)$ and we integrate it with respect to dy:

$$V = \pi \cdot \int_c^d f^2(y)\, dy.$$



EXAMPLE 16 Find the volume V of the solid figure generated by rotating the region between $f(x) = 3x - 1$, the y-axis, and the lines $y = 2$ and $y = 5$ around the y-axis.

Solution To find the volume, we find x in terms of y and integrate the expression with respect to dy:

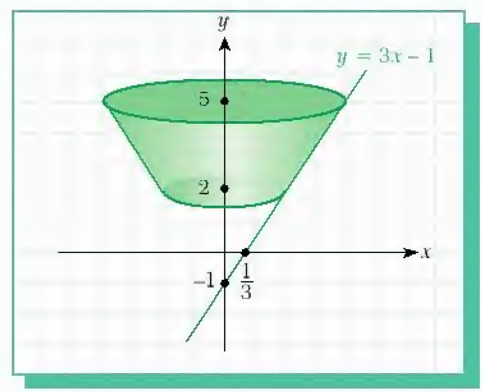
$$y = 3x - 1 \;\Rightarrow\; x = \frac{y+1}{3}$$

$$V = \pi \cdot \int_2^5 \left(\frac{y+1}{3}\right)^2 dy = \pi \cdot \int_2^5 \frac{y^2+2y+1}{9}\, dy$$

$$= \pi \cdot \left(\frac{y^3}{27} + \frac{y^2}{9} + \frac{y}{9}\right)\Big|_2^5$$

$$= \pi \cdot \left(\frac{125}{27} + \frac{25}{9} + \frac{5}{9} - \frac{8}{27} - \frac{4}{9} - \frac{2}{9}\right)$$

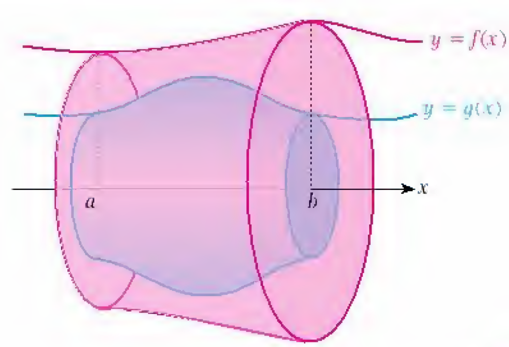
$$= 7\pi.$$



Note

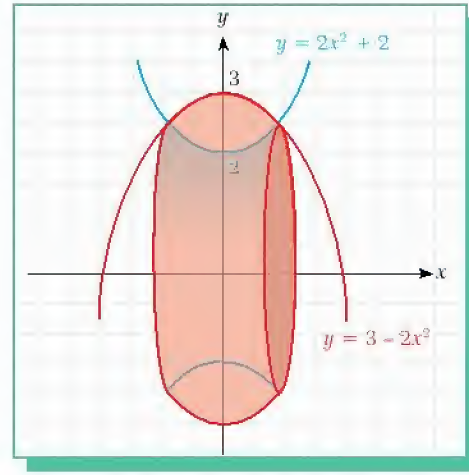
If we rotate the area between two curves $f(x)$ and $g(x)$ on the interval $[a, b]$ then the volume of the solid figure generated is:

$$V = \pi \cdot \int_a^b \left(f^2(x) - g^2(x)\right) dx.$$



EXAMPLE 17 Find the volume V of the solid figure which is generated by rotating the area of the region bounded by the graphs of $y = 2x^2 + 2$ and $y = 3 - 2x^2$ around the x-axis.

Solution $2x^2 + 2 = 3 - 2x^2 \;\Rightarrow\; x = \pm\dfrac{1}{2}$



$$V = \pi\cdot\int_{-1/2}^{1/2}\left[(3-2x^2)^2-(2x^2+2)^2\right]dx$$

$$= \pi\cdot\int_{-1/2}^{1/2}(-20x^2+5)\,dx$$

$$= \pi\cdot\left(-\frac{20x^3}{3}+5x\right)\Bigg|_{-1/2}^{1/2}$$

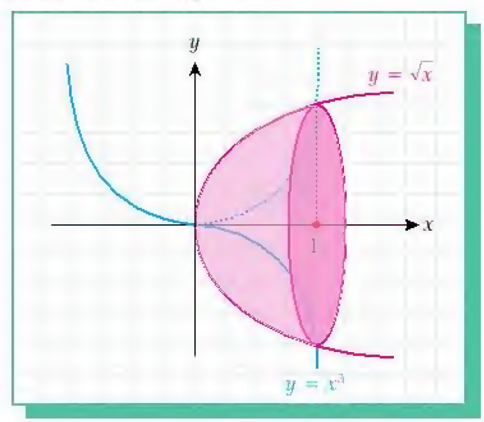
$$= \pi\cdot\left((-\frac{20}{24}+\frac{5}{2})-(\frac{20}{24}-\frac{5}{2})\right)$$

$$= \frac{10\pi}{3}$$

EXAMPLE 18 Find the volume V of the solid figure generated by rotating the area bounded by the graphs of $y = x^3$ and $y = \sqrt{x}$ around the x-axis through 60°.

Solution We can solve the equations simultaneously to find the intersection points:

$x^3 = \sqrt{x},\; x_1 = 0$ and $x_2 = 1.$



$$V = \frac{60}{360}\cdot(\text{total volume})$$

$$= \frac{1}{6}\pi\cdot\int_0^1((\sqrt{x})^2-(x^3)^2)\;dx$$

$$= \frac{1}{6}\cdot\pi\cdot\int_0^1(x-x^6)\;dx$$

$$= \frac{\pi}{6}\cdot\left(\frac{x^2}{2}-\frac{x^7}{7}\right)\Bigg|_0^1$$

$$= \frac{\pi}{6}\cdot\left(\frac{1}{2}-\frac{1}{7}-0\right)$$

$$= \frac{5\pi}{84}$$

EXAMPLE 19 What is the volume V of the solid figure generated by rotating the area bounded by the graphs of $y = e^x$, $x = 1$, $x = 3$ and $y = 1$ around the x-axis?

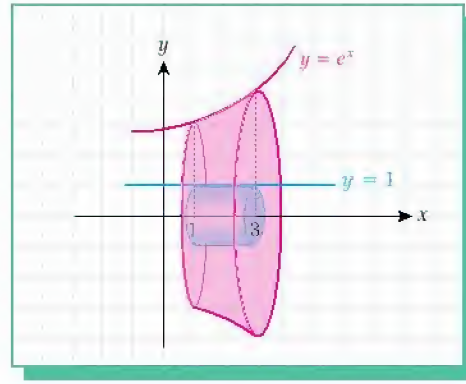
Solution

$$V = \pi \cdot \int_1^3 (e^{2x} - 1)\, dx$$

$$= \pi \cdot \left(\frac{e^{2x}}{2} - x\right) \Big|_1^3$$

$$= \pi \cdot \left(\left(\frac{e^6}{2} - 3\right) - \left(\frac{e^2}{2} - 1\right) \right)$$

$$= \frac{\pi}{2}(e^6 - e^2 - 4)$$



Check Yourself 3

1. Find the volume of the solid figure generated by rotating the area between $y = x^2 - 4$ and the x-axis around the x-axis.
2. Find the volume of the solid figure generated by rotating the area bounded $y = 4x - 1$, the x-axis, and the lines $x = 1$ and $x = 3$ around the x-axis.
3. Find the volume of the solid figure generated by rotating the area bounded by $y = 3x^2 + 2x + 1$, the x-axis, and the lines $x = 0$ and $x = 2$ about the x-axis.
4. Find the volume of the solid figure generated by rotating the area bounded by $y = 2x + 1$, the y-axis, and the lines $y = 2$ and $y = 5$ around the y-axis.
5. Find the volume of the solid figure generated by rotating the area bounded by $y = 2x^2 + x + 2$ and the lines $y = 1$, $x = 1$ and $x = 2$ around the x-axis.
6. Use the definite integral to show that the volume of a sphere with radius R is $\frac{4}{3}\pi R^3$.

Answers

1. $\frac{512\pi}{15}$ 2. $\frac{326\pi}{3}$ 3. $\frac{2134\pi}{15}$ 4. $\frac{21\pi}{4}$ 5. $\frac{349\pi}{5}$
6. Place a circle at the centre of the coordinate plane and use the equation of circle.

B. FINDING THE LENGTH OF A CURVE (*OPTIONAL*)

FINDING LENGTH

The length L of any curve (or line) between the points a and b of a continuous and differentiable function $f(x)$ is:

$$L = \int_a^b \sqrt{1 + (f'(x))^2}\, dx.$$

EXAMPLE 20 Find the length L of the graph of $f(x) = 4(x - 1)^{3/2}$ between $x = 1$ and $x = 2$.

Solution Using the formula we get:

$$f'(x) = 6 \cdot (x - 1)^{1/2}$$

$$L = \int_1^2 \sqrt{1 + (6(x-1)^{1/2})^2}\, dx = \int_1^2 \sqrt{1 + 36(x-1)}\, dx = \int_1^2 \sqrt{36x - 35}\, dx$$

$$u = 36x - 35 \Rightarrow du = 36\, dx \text{ or } dx = \frac{du}{36}$$

$$dx = \frac{1}{36}\int u^{1/2}\, du = \frac{u^{3/2}}{54} = \frac{(36x-35)^{3/2}}{54}\Big|_1^2 = \frac{37\sqrt{37} - 1}{54}.$$

EXAMPLE 21 Find the circumference of a circle with radius 2 units.

Solution Let us assume that the center of the circle is at the origin of a graph, then the equation of the circle is $x^2 + y^2 = 4$.

So $y = \pm\sqrt{4 - x^2}$.

Now let us divide the circle into four parts and find the length of just one part:

$$L = \int_0^2 \sqrt{1 + ((\sqrt{4 - x^2})')^2}\, dx.$$

So the circumference of the circle $= 4 \cdot \int_0^2 \sqrt{1 + \left(-\frac{x}{\sqrt{4 - x^2}}\right)^2}\, dx$

$$= 4 \cdot \int_0^2 \sqrt{1 + \frac{x^2}{4 - x^2}}\, dx$$

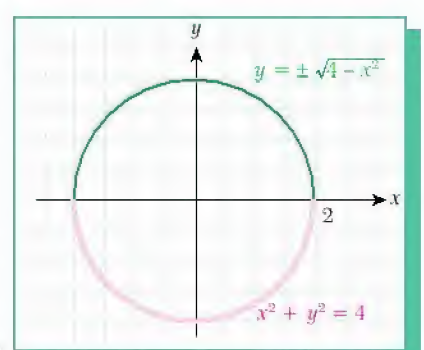
$$= 4 \cdot \int_0^2 \sqrt{\frac{4}{4 - x^2}}\, dx$$

$$= 8 \cdot \int_0^2 \frac{dx}{\sqrt{4 - x^2}}$$

$$= 8 \cdot (\arcsin \frac{x}{2}) \Big|_0^2$$

$$= 8 \cdot (\arcsin 1 - \arcsin 0)$$

$$= 4\pi \text{ units.}$$



C. CALCULATING THE AREA OF A SURFACE OF REVOLUTION (*OPTIONAL*)

FINDING SURFACE OF REVOLUTION

If a function $f(x)$ has a continuous first derivative on $[a, b]$ then the area A of the surface generated by revolving the curve about x-axis is

$$A = 2\pi \cdot \int_a^b f(x) \cdot \sqrt{1+(f'(x))^2}\, dx.$$

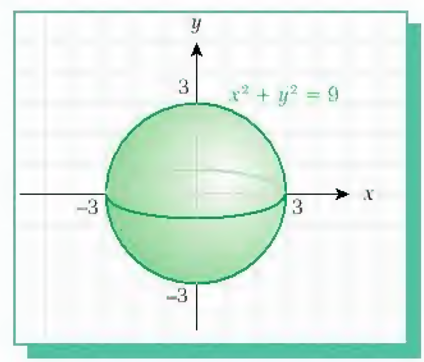
Such a surface is called a surface of revolution.

EXAMPLE 22 Find the surface area of a sphere with radius $r = 3$ cm.

Solution Let us take the circle $x^2 + y^2 = 9 \Rightarrow y = \pm\sqrt{9-x^2}$.

Now let us use the arc between $x = 0$ and $x = 3$ and rotate it. This will give us half of the surface area of the sphere, so we need to multiply the result by 2 to obtain the whole surface area.

$$\text{surface area} = 2 \cdot 2\pi \cdot \int_0^3 \sqrt{9-x^2} \cdot \sqrt{1+((\sqrt{9-x^2})')^2}\, dx$$



$$= 4\pi \cdot \int_0^3 \sqrt{9-x^2} \cdot \sqrt{1+\frac{x^2}{9-x^2}}\, dx$$

$$= 4\pi \cdot \int_0^3 \sqrt{9-x^2} \cdot \frac{3}{\sqrt{9-x^2}}\, dx$$

$$= 4\pi \cdot (3x) \Big|_0^3$$

$$= 36\pi \text{ cm}^2$$

Check Yourself 4

1. Find the length of the curve $y = 2(x+3)^{3/2}$ between $x = 1$ and $x = 3$.
2. Find the length of the curve $y = 4x^2 - x + 1$ on the interval $[0, 1]$
3. Find the length of the curve $y = x^{3/2}$ on the interval $[0, 2]$.
4. Find the area of the surface of revolution which is generated by rotating the curve $y = 2x + 1$ about the x-axis on the interval $[1, 3]$.
5. A parabolic reflector is obtained by rotating the parabola $y = \sqrt{x}$ on the interval $[1, 2]$ about the x-axis. What is the surface area of the reflector?

Answers

1. $\frac{110\sqrt{55}-74\sqrt{37}}{27}$ 2. $\frac{9\sqrt{2}-\ln(\sqrt{2}-1)}{4}$ 3. $\frac{22\sqrt{22}-8}{27}$ 4. $20\pi\sqrt{5}$ 5. $\frac{9\pi}{2}-\frac{5\pi\sqrt{5}}{6}$

Practical Integral Applications

We have seen how to use the definite integral to find the area under a curve, the volume of a solid, and the length of a curve. These results have many practical applications.

For example, Table 1 shows a graph about a cell phone company. The graph shows the number of new users the company hopes to have per month. How many users will there be after five months? The answer is the area under the graph.

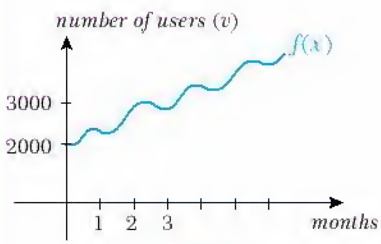
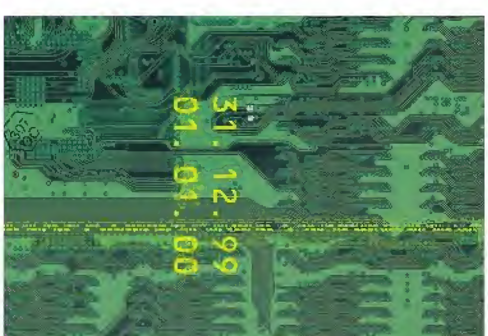


Table 1

The definite integral is also useful in economics and business. Statistics is the branch of mathematics that studies and processes data. Statisticians use tables and graphs to find out about changes over time, for example, changes in a company's income, or changes in the population of a city or country. For example, imagine you are studying the population of an island. You have found that the population increase P over t years is given by $P = 25\sqrt{t} + 20$. The current population is 1200. How many people will be living on the island in thirty years' time? (This problem is left as an exercise for you. Hint: use the definite integral.)



The definite integral also has applications in circuit design, architecture, astronomy and many other fields. Integrals tell us about the dilation of electronic circuits, the curves and surface areas of buildings, and the movements of the stars and planets.

EXERCISES 3.2

A. Calculating the Volume of a Solid of Revolution

1. Find the volume of the solid figure generated by rotating the area of the region bounded by $y = 2x + 5$, $x = 2$ and $x = 3$ around the x-axis.
2. Find the volume of the solid figure generated by rotating the area of the region bounded by $y = x^2 + 1$ and the x-axis on $[0, 1]$ about the x-axis.
3. Find the volume of the solid figure generated by rotating the area of the region bounded by $y = x^3 - 1$ and the x-axis on $[1, 2]$ around the x-axis.
4. Find the volume of the solid figure generated by rotating the area of the region bounded by $y = 3x + 1$, the x-axis, and the lines $x = 1$ and $x = 3$ about the x-axis.
5. Find the volume of the solid figure generated by rotating the area of the region bounded by $y = 1 - x^2$ and the x-axis around the x-axis.
6. Find the volume of the solid figure generated by rotating the area of the region bounded by $y = x^2 - 4$ and the x-axis about the x-axis.
7. Find the volume of the solid figure generated by rotating the area of the region bounded by $y = 2x - 1$, the y-axis, and the lines $y = 1$, $y = 2$ 0 about the x-axis.
8. Find the volume of the solid figure generated by rotating the area of the region bounded by $y = 2x^2 - 1$, the y-axis, and the lines $y = 0$ and $y = 3$ about the y-axis.
9. Find the volume of the solid figure generated by rotating the area of the region bounded by $y = x^2 + 4$ and $y = 2$ on the interval $[1, 3]$ about the x-axis.
10. Find the volume of the solid figure generated by rotating the area of the region bounded by $y = x^2 - 4$ and $y = 3x + 6$ and x-axis about the x-axis.
11. Find the volume of the solid figure generated by rotating the area of the region bounded by $y = 4x - 1$, and the x-axis on $[0, 3]$ about y-axis.
12. Find the volume of the solid figure generated by rotating the area of the region bounded by $y = e^x$, $x = 2$ and the x and y axes about the x-axis.
13. Find the volume of the solid figure generated by rotating the area of the region bounded by $y = \sin x$, $x = 0$, $x = \pi$ and x-axis about the x-axis.
14. Find the volume of the solid figure generated by rotating the area of the region bounded by $y = \cos 2x$, $x = \frac{\pi}{2}$, and x-axis about the x-axis.
15. Find the volume of the solid figure generated by rotating the area of the region bounded by $y = x^2$ and $y = x$ about the x-axis.
16. Find the volume of the solid figure generated by rotating the area of the region bounded by $y = x^2$ and $y = x$ about the y-axis.
17. Find the volume of the solid figure generated by rotating the area of the region bounded by $y^2 = x + 4$, $x = 2$ and $y = 2$ about the x-axis.
18. Find the volume of the solid figure generated by rotating the area of the region bounded by $y = 2x^2 + 3x - 1$ and $y = x^2 + x - 2$ about the x-axis on $[1, 3]$.
19. Find the volume of a cone with radius $r = 3$ cm and altitude 4 cm by using integration.
20. Find the volume of the solid figure generated by rotating the area of the region bounded by $y = x^2 + 1$ and $y = 3x - 1$ about the x-axis.

21. Find the volume of the solid figure generated by rotating the area of the region between $y = \tan x$ and the x-axis on the interval $\left[0, \frac{\pi}{3}\right]$ about the x-axis through 180°.

22. Find the volume of the solid figure generated by rotating the area of the region bounded by $y = 5 - x^2$ and $y = x^2 + 3$ about the x-axis.

23. Find the volume of the solid figure generated by rotating the area of the region bounded by $f(x) = -x^2$ and $g(x) = x^2 - 3$ about the x-axis.

24. ✪ Find the volume of the solid figure generated by rotating the area of the region bounded by $f(x) = \frac{1}{x}$, the x-axis, $x = 1$ and $x = 3$ about the y-axis.

25. Find the volume of the solid figure generated by rotating the area of the region bounded by $f(x) = \frac{1}{x}$, the x-axis, $x = 1$ and $x = 3$ about the x-axis.

26. Find the volume of the solid figure generated by rotating the area of the region bounded by $y = x^2 + x + 1$, $x = 1$, $x = 2$ and the x-axis about the x-axis through 90°.

27. Find the volume of the solid figure generated by rotating the area of the region bounded by $y = e^x$, $x = 1$, $x = 2$ and the x-axis about the x-axis through 120°.

28. ✪✪ Find the volume of the solid figure generated by rotating the area of the region bounded by $y = 5 - x^2$, $y = 2$, $x = 0$, $x = 1$ about $y = 1$.

29. Find the volume of the solid figure generated by rotating the area of the region bounded by $y = 1 - x^2$, the x-axis, $x = 1$ and $x = 3$ about the y-axis.

30. Find the volume of the solid figure generated by rotating the area of the region bounded by $y = x^3$, the x-axis, $x = 0$ and $x = 2$ about the y-axis.

B. Finding the Length of a Curve

31. Find the length of the graph $y = 3x + 1$ between $x = 0$ and $x = 4$.

32. Find the length of the curve $y = 2 \cdot (x - 1)^{3/2}$ on the interval $[1, 2]$.

33. Use integration to find the circumference of a circle with radius 5 cm.

34. Find the length of the parabola $y^2 = x$ on the interval $[0, 1]$.

35. Find the length of the graph $y = x^{3/2}$ over $[0, 1]$.

36. ✪ Find the length of the curve $y = x^2 - 1$ between $x = 0$ and $x = 1$.

C. Calculating the Area of a Surface of Revolution

37. Use integration to find the surface area of a sphere with radius 2 cm.

38. Find the surface area of the solid figure generated by revolving the parabola $y = x^2$ around the x-axis on the interval $[0, 1]$.

39. Find the surface area of the solid generated by rotating the curve $y = \frac{2x^{3/2}}{3}$ on $[1, 2]$ about the x-axis.

40. Calculate the surface area of the solid obtained by rotating the graph of $f(x) = \sqrt{x}$ on the interval $[0, 1]$ about the y-axis.

CHAPTER SUMMARY

- Area

1. If $f : [a, b] \to R$ is a continuous, positively defined function ($f(x) \geq 0$) then the area A of the region between $y = f(x)$ and the x-axis on the interval $[a, b]$ is

$$A = \int_a^b f(x)\,dx.$$

2. If $y = f(x)$ is any continuous function in the interval $[a, b]$ then the area between $y = f(x)$ and the x-axis is

$$\int_a^b |f(x)|\,dx.$$

3. The area A between $y = f(x)$, the y-axis and the lines $y = m$ and $y = n$ is

$$A = \int_m^n f(y)\,dy.$$

4. Let $f(x)$ and $g(x)$ be two curves. Then the area A between $f(x)$ and $g(x)$ on the interval $[a, b]$ is

$$A = \int_a^b [f(x) - g(x)]\,dx.$$

- Volume of a Solid of Revolution

1. Let $f(x)$ be a continuous function defined on $[a, b]$. Then the volume V of the solid obtained by rotating the area between $f(x)$ and the x-axis on $[a, b]$ is

$$V = \pi \cdot \int_a^b f^2(x)\,dx.$$

2. If we rotate the figure around the y-axis we use $x = f(y)$ and integrate with respect to dy:

$$V = \pi \cdot \int_c^d f^2(y)\,dy.$$

3. If we rotate the area between two curves $f(x)$ and $g(x)$ on the interval $[a, b]$ then the volume V of the solid figure is:

$$V = \pi \cdot \int_a^b f^2(x) - g^2(x)\,dx.$$

- Length of a Curve

The length L of any curve (or line) between the points a and b for a continuous and differentiable function $f(x)$ is

$$L = \int_a^b \sqrt{1 + (f'(x))^2}\,dx.$$

- Area of a Surface of Revolution

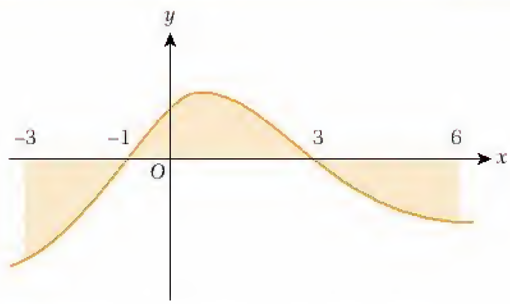
If a function $f(x)$ has a continuous first derivative on $[a, b]$ then the area A of the surface generated by rotating the curve about the x-axis is

$$A = 2\pi \cdot \int_a^b f(x) \cdot \sqrt{1 + (f'(x))^2}\,dx.$$

Concept Check

- How can we use integration to find the area under a curve?
- How can we find the area between the graph of a function and the x-axis if the function is below the x-axis?
- How can we find the area between two curves if they intersect at a countable number of points?
- How can we find the area between the graph of a function and the y-axis?
- How can we find the volume of a function rotated around an axis by using integration?
- How can we find the length of a curve on a closed interval $[a, b]$?
- How can we use integration to find the area of revolution of a surface?

1.



In the figure, $\int_{-1}^{3} f(x)\,dx = 12 \text{ cm}^2$ and $\int_{-3}^{6} f(x)\,dx = 5 \text{ cm}^2$.

What is the total area of the shaded region in cm^2?

A) 7 B) 12 C) 17 D) 19 E) 21

2. What is the area of the region bounded by the graphs of $y = 2x - 3$, the x-axis, and the lines $x = 2$ and $x = 4$?

A) 6 B) 9 C) 10 D) 16 E) 21

3. What is the area of the region bounded by the graphs of $y = -x + 5$, the x-axis, and the lines $x = 3$ and $x = 5$?

A) 27 B) 13 C) 7 D) 3 E) 2

4. What is the area of the region bounded by the graph of $y = x^2 - 9$ and the x-axis?

A) 27 B) 36 C) 28 D) 40 E) 49

5. What is the area of the region bounded by the graphs of $y = x^2 - 3x + 2$, the x-axis, and the lines $x = 1$ and $x = 3$?

A) $\frac{1}{6}$ B) $\frac{5}{6}$ C) 1 D) $\frac{7}{12}$ E) $\frac{21}{5}$

6. What is the area of the region bounded by the graph of $y = x^3$, the y-axis, and the line $y = 8$?

A) 12 B) $\frac{8}{3}$ C) $\frac{16}{3}$ D) 8 E) 17

7. What is the area of the region bounded by the graph of $y = x^2 - 7x + 10$ and the line $y = x + 3$?

A) 33 B) 27 C) 18 D) 36 E) 42

8. What is the area of the region bounded the graphs of $y = x^2 - x + 1$ and $y = -x^2 + 2x + 3$?

A) $\frac{146}{7}$ B) $\frac{68}{3}$ C) $\frac{125}{24}$ D) $\frac{17}{2}$ E) 49

9. What is the area of the region bounded by the graph of $y = \sin 2x$ and the x-axis on the interval $[0, \pi]$?

A) 2 B) 4 C) 5 D) 3π E) 4π

10. What is the area of the region bounded by the graph of $y = 3x - 4$, the y-axis, and the lines $y = 1$ and $y = 4$?

A) $\frac{7}{3}$ B) $\frac{13}{2}$ C) $\frac{8}{5}$ D) $\frac{43}{7}$ E) 10

11. What is the volume of the solid figure generated by rotating the area between $y = x$ and the x-axis around the x-axis on the interval $[0, 3]$?

A) $\frac{\pi}{3}$ B) 27π C) $\frac{27\pi}{2}$ D) 27 E) 9π

12. What is the volume of the solid figure generated by rotating the area between $y = x^2 + 1$, the x-axis, and the lines $x = 0$ and $x = 1$ around the x-axis?

A) π B) $\frac{19\pi}{2}$ C) 4π D) $\frac{28\pi}{15}$ E) $\frac{17\pi}{3}$

13. What is the volume of the solid figure generated by rotating the area between the graphs of $y = x^2 + 2$ and $y = 3$ around the x-axis?

A) $\frac{27\pi}{5}$ B) $\frac{104\pi}{15}$ C) $\frac{48\pi}{5}$ D) $\frac{163\pi}{15}$ E) $\frac{32\pi}{3}$

14. What is the volume of the solid figure generated by rotating the area between the graphs of $y = 3x + 2$ and $y = x + 8$ around the x-axis on the interval $[0, 1]$?

A) 65π B) 27π C) $\frac{178\pi}{3}$ D) $\frac{147\pi}{3}$ E) $\frac{63\pi}{3}$

15. What is the volume of the solid figure generated by rotating the area of the region between $y = 3x - 1$, the y-axis, and the lines $y = 1$ and $y = 2$ around the y-axis ?

A) $\frac{19\pi}{27}$ B) $\frac{\pi}{3}$ C) $\frac{47\pi}{9}$ D) $\frac{17\pi}{5}$ E) $\frac{2\pi}{3}$

16. What is the volume of the solid figure generated by rotating the area of the region between the graph of $y = 2x - 1$ and the x-axis through 180° on the interval $[0, 2]$?

A) $\frac{21\pi}{3}$ B) $\frac{21\pi}{4}$ C) $\frac{14\pi}{3}$ D) $\frac{7\pi}{3}$ E) 5π

CHAPTER REVIEW TEST 3B

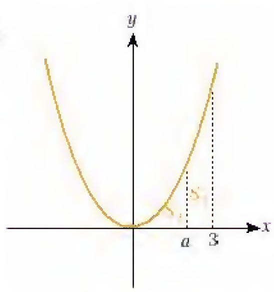
1. What is the area of the region bounded by the graphs of $y = 3x - 1$ and the lines $x = 0$ and $y = 2$?

A) $\frac{3}{7}$ B) $\frac{3}{2}$ C) $\frac{2}{3}$ D) 1 E) $\frac{7}{3}$

2. What is the area of the region bounded by the graph of $y = 3x^2 + 4$, the x-axis, and the lines $x = 1$ and $x = 3$?

A) 34 B) 29 C) 21 D) 16 E) 9

3. The figure shows the graph of $f(x) = x^2$. Given $2S_1 = S_2$, find a.



A) 2 B) $2\sqrt[3]{2}$ C) $\sqrt[3]{9}$ D) $\sqrt[3]{8}$ E) $\sqrt{13}$

4. What is the area of the region bounded by the graph of $y = 5x + 1$ and the x and y-axes?

A) 1 B) 3 C) 0.3 D) 0.2 E) 0.1

5. What is the area of the region bounded by the graphs of $y = 2x + 1$ and $y = 5 - 2x$ and the x-axis?

A) $\frac{1}{6}$ B) $\frac{5}{6}$ C) $\frac{9}{2}$ D) $\frac{7}{12}$ E) $\frac{21}{5}$

6. What is the area of the region bounded by the graph of $y = x^3 - 4x$ and the x-axis?

A) 12 B) $\frac{8}{3}$ C) $\frac{16}{3}$ D) 8 E) 16

7. What is the area of the region between the graphs of $y = x^2 - 6x$ and $y = -9$ and the y-axis?

A) 32 B) 27 C) 18 D) 9 E) 4

8. What is the area of the region between the graphs of $y = x^2 - x + 1$ and $y = -x^2 - 4x + 3$?

A) $\frac{145}{8}$ B) $\frac{125}{24}$ C) $\frac{127}{25}$ D) $\frac{17}{2}$ E) 49

9. What is the area of the region bounded by the graphs of $y = \cos 2x$ and $x = \pi$ and the x-axis?

A) 2 B) 4 C) 5 D) 3π E) 4π

10. What is the area of the region bounded by the graphs of $y = \cos 2x$ and $y = \sin x$ on the interval $[0, \pi]$?

A) $3\sqrt{3} - 2$ B) $\sqrt{3} - 2$ C) $12\pi - 2$ D) $6\sqrt{3} + 2$ E) $6\sqrt{3} - 2$

11. What is the area of the region bounded by the graphs of $x = y^2 + 1$, y-axis, $y = 1$ and $y = 3$?

A) $\frac{17}{3}$ B) $\frac{32}{3}$ C) 9 D) $\frac{28}{3}$ E) $\frac{17}{6}$

12. What is the volume of the solid generated by rotating the area between the graphs of $y = 3x - 1$, $x = 2$ and $x = 3$ around the x-axis?

A) $\frac{27\pi}{5}$ B) 16π C) 22π D) 42π E) 43π

13. What is the volume of the solid figure generated by rotating the area between $y = x^2 - 2x$ and the x-axis around the x-axis ?

A) 2π B) $\frac{19\pi}{3}$ C) 7π D) $\frac{28\pi}{15}$ E) $\frac{16\pi}{15}$

14. What is the volume of the solid figure generated by rotating the area between the graphs of $y = 5x - 1$, $x = 1$, $x = 2$ and the x-axis around the y-axis?.

A) $\frac{35\pi}{3}$ B) 27π C) $\frac{71\pi}{3}$ D) $\frac{147\pi}{3}$ E) $\frac{63\pi}{2}$

15. What is the volume of the solid figure generated by rotating the area between the graphs of $y = x^2 - 3x - 4$, $x = 2$, $x = 5$ and the x-axis around the x-axis?

A) $\frac{197\pi}{27}$ B) $\frac{28\pi}{3}$ C) $\frac{441\pi}{10}$ D) $\frac{17\pi}{3}$ E) $\frac{2\pi}{3}$

16. What is the volume of the solid figure generated by rotating the area between the graphs of $y = 2x - 1$, $x = 1$, $x = 2$ and the x-axis through 180° around y-axis?

A) $\frac{64\pi}{3}$ B) $\frac{41\pi}{12}$ C) $\frac{14\pi}{5}$ D) $\frac{96\pi}{5}$ E) 5π

ANSWERS TO EXERCISES

EXERCISES 1.1

1. a. $u + c$ b. $z + c$ c. $\cos x + c$ d. $x^3 + 3x^2 + c$ 2. 26 3. $2 + \dfrac{5}{x}$ 4. $\dfrac{31}{8}$ 5. $\dfrac{97}{3}$ 6. a. $\sin x + x^4 - e^{2x} + c$

b. $\dfrac{5x^4}{4} - x^3 + 5x + c$ c. $5\arctan x + c_1 = -5\operatorname{arccot} x + c_2$ d. $\dfrac{7x^9}{9} + c$ e. $\dfrac{1}{4}\sin 4x + c$ f. $-7\cos x + c$ g. $\dfrac{5}{3}e^{3x} + c$

h. $\dfrac{2}{3}\sin 3x - 4\cos x - \dfrac{4}{5}e^{5x} + c$ 7. a. $\dfrac{x^6}{6} + c$ b. $4x + c$ c. $-\dfrac{1}{2x^2} + c$ d. $-\dfrac{1}{4x^4} + c$ e. $\dfrac{3x^8}{8} + c$ f. $\dfrac{1}{8x^8} - \dfrac{1}{5x^5} - \dfrac{1}{2x^2} + c$

g. $4e^{2x} + c$ h. $x^3 + 2x^2 - x + c$ 8. a. $-\dfrac{2\sqrt{x}}{x} + c$ b. $3\ln|x| + c$ c. $-\dfrac{1}{x} + \ln|x| + x + \dfrac{x^2}{2} + c$ d. $\sin x - \cos x + c$

e. $\ln|x| + c$ f. $5\ln|x + 1| + c$ g. $\ln|x - 1| + c$ h. $2x^2 + 3x - 4\ln|x| - \dfrac{1}{x} + c$ 9. a. $\dfrac{e^{2x}}{2} + c$ b. $\dfrac{e^{5x}}{5} + c$

c. $\dfrac{3e^{2x}}{2} + c$ d. $\dfrac{5e^{7x+2}}{7} + c$ e. $7e^{x-2} + c$ f. $\dfrac{4^x}{\ln 2} + c$ g. $\dfrac{5^x}{\ln 5} + c$ h. $\dfrac{6^{2x-1}}{2\ln 6} + c$ i. $\dfrac{4^{3x-4}}{6\ln 2} + c$ j. $\dfrac{3^{3x-1}}{\ln 3} + c$

k. $\dfrac{10^{x+1}}{\ln 10} + c$ l. $\dfrac{2 \cdot 3^{2x-1}}{\ln 3} + c$ 10. a. $-\dfrac{\cos 4x}{4} + c$ b. $\dfrac{\sin 5x}{5} + c$ c. $4\tan x + c$ d. $-\dfrac{5\cot 2x}{2} + c$ e. $\tan 4x + c$

f. $\tan x - x + c$ g. $x - \cot x + c$ h. $3\arcsin x + c_1 = -3\arccos x + c_2$ i. $4\arctan x + c_1 = -4\operatorname{arccot} x + c_2$

j. $x + 4\arctan x + c_1 = x - \operatorname{arccot} x + c_2$ k. $\dfrac{5\sin(8x - 4)}{8} + c$ l. $\dfrac{\arcsin 2x}{2} + c_1 = -\dfrac{\arccos 2x}{2} + c_2$

m. $\dfrac{5}{3}\arctan 3x + c_1 = -\dfrac{5}{3}\operatorname{arccot} 3x + c_2$ n. $\dfrac{x}{2} - \dfrac{\sin 2x}{4} + c$ o. $-\cot x - x + c$ p. $\tan x - 2x + c$

EXERCISES 1.2

1. a. $\ln|x - 3| + c$ b. $-\ln|x| + c$ c. $\ln|x| + c$ d. $\dfrac{5}{3}\ln|3x + 1| + c$ e. $-\dfrac{\cos(4x+1)}{4} + c$ f. $\dfrac{(1 + x^2 + x^3)^9}{9} + c$ g. $-\dfrac{(1 - x^2)^8}{16} + c$

h. $\dfrac{\sin(x^2 - 5)}{2} + c$ i. $\dfrac{\arcsin 4x}{4} + c_1 = -\dfrac{\arccos 4x}{4} + c_2$ j. $e^{\sin x} + c$ k. $\arctan(\sin x) + c_1 = -\operatorname{arccot}(\sin x) + c_2$

l. $\dfrac{(1 + x^2)^{\frac{3}{2}}}{3} + c$ m. $\dfrac{(x^4 + x^2)^2}{4} + c$ 2. a. $\dfrac{\sin x^2}{2} + c$ b. $-\dfrac{\cos(5x^2 + 7)}{10} + c$ c. $\dfrac{\ln^2 x}{2} + c$ d. $\ln|\sin x| + c$

e. $\frac{1}{9(1-3x)^3}+c$ f. $\ln(e^x+e^{-x})+c$ g. $\ln|e^x-3|+c$ h. $-\frac{5\cos^{\frac{4}{5}}x}{4}+c$ i. $\frac{(10x+4)\sqrt{5x-1}}{75}+c$ 3. a. xe^x-e^x+c

b. $e^x(x^2-2x+2)+c$ c. $e^x(x^3-3x^2+6x-6)+c$ d. $\sin x-x\cos x+c$ e. $-e^{-x}(x^2+2x+2)+c$

f. $x\arccos x-\sqrt{1-x^2}+c$ g. $(x+5)\ln|x+5|-x+c$ h. $x\log x-\frac{x}{\ln 10}+c$ i. $x\,\mathrm{arccot}\,x+\frac{1}{2}\ln(1+x^2)+c$

j. $\frac{x}{2}(\cos(\ln|x|)+\sin(\ln|x|))+c$ k. $-e^{2x}\cdot(\frac{\sin x\cdot\cos x}{4}-\frac{\sin^2 x}{4}-\frac{1}{8})+c$ 4. a. $3\ln|2x+1|+c$ b. $-\frac{1}{(3x+1)^3}+c$

c. $\frac{x^2}{2}+2\ln|x|-\frac{1}{x}+c$ d. $\ln|x^2+x-1|+c$ e. $\frac{11}{6}\ln|3x-1|+\frac{1}{2}\ln|x-1|+c$ f. $4x+\frac{5}{2}\ln|x^2+1|+c$ g. $-\frac{1}{x+2}+c$

h. $-\frac{1}{2(x+1)^2}+c$ i. $5\ln|x+2|-2\ln|x+1|+c$ j. $\frac{1}{2}\ln|x^2-2x-3|+c$ k. $\frac{1}{2}(\ln|x-1|+3\ln|x+1|)+c$

l. $\frac{1}{2}\ln|\frac{x+3}{x+5}|+c$ m. $\ln|x-1|-\frac{5}{x-1}-\frac{3}{2(x-1)^2}+c$ n. $-\ln|1+x|-\frac{2}{1+x}+c$ o. $7\ln|x-1|+2\ln|x^2+x+2|+c$

p. $\frac{5}{3}\ln|x-1|-\frac{1}{3}\ln(x^2+x+1)+c$ q. $\frac{3x^2}{2}-4x+\ln|x^3-4x^2+1|+c$ r. $\frac{7\sqrt{2}}{6}\arctan(\frac{x\sqrt{2}}{2})-\frac{\ln(x^2+2)}{6}+\frac{\ln|x+1|}{3}+c$

s. $\frac{\sqrt{3}}{6}\ln|\frac{x-\sqrt{3}-2}{x+\sqrt{3}-2}|+c$ t. $\frac{\sqrt{3}}{6}\arctan(\frac{x^2\sqrt{3}}{3})+c$ 5. a. $\frac{2(5x-1)^{\frac{3}{2}}}{15}+c$ b. $-\frac{2(1-x)^{\frac{3}{2}}}{3}+c$ c. $\frac{(1+x^2)^{\frac{3}{2}}}{3}+c$

d. $\frac{3(x+1)^{\frac{4}{3}}}{4}+c$ e. $-\sqrt{1-x^2}+c$ f. $\sqrt{5x^2+3}+c$ g. $\frac{5(1+x)^{\frac{6}{5}}}{6}+c$ h. $\frac{(1+x^3)^{\frac{2}{3}}}{2}+c$ i. $x+6\sqrt{x-2}+c$

6. a. $\frac{x}{2}\sqrt{1-4x^2}-\frac{\arccos 2x}{4}+c$ b. $\arcsin x+c$ c. $\ln|\sqrt{x^2+1}+x|+c$ d. $\frac{\sqrt{16x^2+1}}{16}+c$ e. $\sqrt{16x^2-9}-3\,\mathrm{arcsec}\frac{4x}{3}+c$

f. $\frac{x\sqrt{16-9x^2}}{2}-\frac{8}{3}\arccos(\frac{3x}{4})+c$ g. $\frac{\ln(\sqrt{9x^2+1}+3x)}{6}+\frac{x\sqrt{9x^2+1}}{2}+c$ h. $\frac{x}{2}\sqrt{x^2-9}-\frac{9}{2}\ln(\sqrt{x^2-9}+x)+c$

i. $\ln|\sqrt{x^2-9}+x|+c$ 7. a. $\frac{\sin^3 x}{3}+c$ b. $\frac{\sin^2 x}{2}+c$ c. $\frac{\sin^6 x}{6}-\frac{\sin^8 x}{8}+c$ d. $-\frac{\cos^3 x}{3}+c$ e. $\frac{\cos^8 x}{8}-\frac{\cos^6 x}{6}+c$

f. $\frac{\sin^5 x}{5}-\frac{\sin^7 x}{7}+c$ g. $\frac{\cos^7 x}{7}-\frac{\cos^5 x}{5}+c$ h. $\frac{\sin 7x}{14}+\frac{\sin x}{2}+c$ i. $\frac{\cos^{10} x}{5}-\frac{\cos^8 x}{8}-\frac{\cos^{12} x}{12}+c$

j. $\frac{\cos x}{2}-\frac{\cos 7x}{14}+c$ k. $\frac{\sin 2x}{4}-\frac{\sin 12x}{24}+c$ l. $\frac{\cos 5x}{10}-\frac{\cos 11x}{22}+c$ m. $-\frac{\cos 6x}{12}-\frac{\cos 2x}{4}+c$ n. $\frac{\cos 4x}{8}-\frac{\cos 6x}{12}+c$

o. $\frac{\sin 5x}{10}+\frac{\sin 3x}{6}+c$ p. $\frac{\sin^3 4x}{6144}+\frac{3\sin 8x}{8192}-\frac{\sin 4x}{512}+\frac{5x}{1024}+c$ 8. a. $-\ln(3+\cos x)+c$

b. $\frac{1}{2}\ln|\tan\frac{x}{2}|-\frac{1}{4}\tan^2\frac{x}{2}+c$ c. $\ln|\tan\frac{x}{2}|+c$ d. $6\ln|\tan\frac{x}{2}|-3\ln|\tan^2\frac{x}{2}+1|+c$ e. $2\cdot\dfrac{1+\tan\frac{x}{2}}{1-\tan\frac{x}{2}}-x+c$

EXERCISES 2.1

1. 4.5 2. integral = 16, area $\frac{52}{3}$ 3. area = 16, integral = 8 4. a. $\frac{19}{3}$ b. 36 c. 64 d. $\frac{1}{2}$ e. 1 f. 0 g. $\frac{20}{3}$ h. $\frac{5a^4}{6}$ i. $\frac{74}{3}$ 5. a. 0 b. 0 c. 37 d. $\frac{3787}{12}$ e. $\frac{10537}{15}+\ln 4$ f. $\frac{2}{3}$ g. $\frac{e^3+3e-1}{3}$ 6. –7 7. –3

8. a. $\cos x$ b. $\frac{1+2x}{x^4}$ c. $2x\sin x^2$ d. $3x^5+12x^2-x-4$ 9. a. $\frac{7}{2}$ b. $\frac{9}{4}$ c. 2 d. 0

EXERCISES 2.2

1. a. $\frac{17}{2}$ b. 34 c. $\frac{41}{2}$ d. 40 e. $\frac{109}{3}$ f. $\frac{569}{6}$ g. 2 h. $\frac{17}{4}$ i. 25 2. a. 6 b. 2 c. 8 d. 3 e. 4 f. –1

3. a. 3 b. $-\frac{3}{2}$ c. $\frac{9}{2}$ d. 1 e. 10 f. 0 g. 3 h. $\frac{191}{8}$

EXERCISES 3.1

1. $\frac{32}{3}$ 2. $\frac{9}{2}$ 3. $\frac{13}{2}$ 4. $\frac{57}{2}$ 5. 24 6. $\frac{157}{3}$ 7. 2 8. $\frac{11}{4}$ 9. $4\sqrt{3}-\frac{10}{3}$ 10. 1 11. $\frac{4}{3}$ 12. $\frac{26}{3}$ 13. $\frac{32}{3}$

14. $\frac{1}{6}$ 15. $\frac{19}{3}$ 16. $\frac{5}{2}$ 17. $\frac{8}{3}$ 18. $\frac{5}{12}$ 19. $\frac{8}{3}$ 20. $\frac{343}{24}$ 21. $\frac{125}{6}$ 22. $\frac{32}{3}$ 23. $\frac{71}{6}$ 24. $\frac{32}{3}$ 25. $\frac{4}{3}$

26. $\frac{1}{3}$ 27. $2\sqrt{2}$ 28. $\frac{5\sqrt{3}-6}{4}$ 29. $\frac{3-\sqrt{3}+4\sqrt{2}}{6}$ 30. $\frac{2\sqrt{3}-1}{2}$ 31. 4 32. $\frac{5}{2}$ 33. $\frac{3}{2}$ 34. $\frac{256}{27}$ 35. –6 36. 2

37. 3 38. e^2 39. 15 40. 13 41. $\frac{15}{2}$ 42. $\frac{8}{3}$ 43. $6\sqrt[3]{2}-4$ 44. $\frac{5}{6}$ 45. $\frac{256}{27}$ 46. $\frac{1}{2}$ 47. $\frac{13\sqrt{13}}{6}$

48. $\frac{5}{3}$ 49. $\frac{\sqrt{3}+4\sqrt{2}-6}{4}$ 50. $\frac{e-2}{2}$ 51. $\frac{19-16\sqrt{3}}{24}$ 52. $\frac{4\sqrt{2}-1}{3}$

EXERCISES 3.2

1. $\frac{301\pi}{3}$ 2. $\frac{28\pi}{15}$ 3. $\frac{163\pi}{14}$ 4. 104π 5. $\frac{16\pi}{15}$ 6. $\frac{512\pi}{15}$ 7. $\frac{23\pi}{6}$ 8. $\frac{15\pi}{4}$ 9. $\frac{2126\pi}{15}$ 10. $\frac{3372\pi}{5}$ 11. $\frac{21\pi}{16}$

12. $\frac{\pi}{2}(e^4-1)$ 13. $\frac{\pi^2}{2}$ 14. $\frac{\pi^2}{4}$ 15. $\frac{2\pi}{15}$ 16. $\frac{\pi}{6}$ 17. 2π 18. $\frac{6008\pi}{15}$ 19. 12π 20. $\frac{17\pi}{15}$ 21. $\frac{\pi}{6}(3\sqrt{3}-\pi)$

22. $\frac{64\pi}{3}$ 23. $6\sqrt{6}\pi$ 24. 4π 25. $\frac{2\pi}{3}$ 26. $\frac{247\pi}{40}$ 27. $\frac{\pi}{6}(e^4-e^2)$ 28. $\frac{36\pi}{5}$ 29. 96π 30. $\frac{64\pi}{5}$ 31. $4\sqrt{10}$

32. $\frac{20\sqrt{10}-2}{27}$ 33. 10π 34. $\frac{\ln(\sqrt{5}+2)+2\sqrt{5}}{4}$ 35. $\frac{13\sqrt{13}-8}{27}$ 36. $\frac{\ln(\sqrt{5}+2)+2\sqrt{5}}{4}$ 37. 16π

38. $\frac{\pi(18\sqrt{5}-\ln(\sqrt{5}+2))}{32}$ 39. $\frac{\pi(3\ln(\sqrt{6}-\sqrt{3}-\sqrt{2}+2)+33\sqrt{6}-7\sqrt{2})}{18}$ 40. $\frac{\pi(18\sqrt{5}-\ln(\sqrt{5}+2))}{32}$

ANSWERS TO TESTS

TEST 1A

1.	E	9.	A
2.	B	10.	D
3.	E	11.	B
4.	C	12.	B
5.	E	13.	C
6.	A	14.	E
7.	B	15.	D
8.	A	16.	B

TEST 1B

1.	D	9.	D
2.	B	10.	A
3.	C	11.	B
4.	D	12.	D
5.	B	13.	A
6.	B	14.	A
7.	C	15.	A
8.	C	16.	D

TEST 1C

1.	C	9.	B
2.	A	10.	C
3.	B	11.	E
4.	C	12.	C
5.	C	13.	C
6.	D	14.	B
7.	D	15.	E
8.	E	16.	E

TEST 2A

1.	D	9.	E
2.	A	10.	E
3.	C	11.	A
4.	B	12.	D
5.	B	13.	B
6.	E	14.	C
7.	D	15.	A
8.	C	16.	D

TEST 2B

1.	E	9.	C
2.	A	10.	B
3.	E	11.	D
4.	D	12.	E
5.	B	13.	B
6.	C	14.	A
7.	A	15.	C
8.	D	16.	C

TEST 3A

1.	D	9.	A
2.	A	10.	B
3.	E	11.	E
4.	B	12.	D
5.	C	13.	B
6.	A	14.	C
7.	D	15.	A
8.	C	16.	D

TEST 3B

1.	B	9.	A
2.	A	10.	A
3.	C	11.	B
4.	E	12.	E
5.	C	13.	E
6.	D	14.	C
7.	D	15.	C
8.	B	16.	B

GLOSSARY

A

absolute value function: a function which is always defined positively: if $f(x) \geq 0$ then $|f(x)| = f(x)$, and if $f(x) < 0$ then $|f(x)| = -f(x)$.

antiderivative: a function $F(x) + c$ for which $F'(x) = f(x)$.

B

boundary: a curve or a point which limits a region or a line.

C

constant of integration: the constant term c which must be added when calculating an indefinite integral.

continuous function: a function whose graph is a continuous line, with no breaks.

D

definite integral: the area between the graph of a continuous function $f(x)$ on an interval $[a, b]$ and the x-axis.

derivative: the rate of change of a function at a given point.

differentiable function: a function which has a derivative at a given point.

differential: the expression dx which shows the variable in an integral.

differentiation: the process of finding the derivative of a function.

discontinuity: a point at which a mathematical function is not continuous.

F

floor function: a function which gives the greatest integer number which is smaller than the value of a given function.

fundamental theorem of calculus: the theorem that we use to find the definite integral of a function.

I

indefinite integral: the set of all the antiderivatives of a function.

integrable function: a function which has an integral on a given interval.

integral: a mathematical term that can be interpreted as the area under a graph or as a generalization of this area.

integral sign: the sign $\int$ that we use to show the integral of a function.

integrand: the algebraic expression under the integral sign.

integration: the process of finding the integral of a function.

integration by parts: a technique for finding an integral of the form $\int u \cdot v' \, dx$ by expanding the differential of a product of functions $d(uv)$ and expressing the original integral in terms of a known integral.

interval: the set of all real numbers between two known numbers a and b, written $[a, b]$.

inverse conversion formulas: the formulas for writing the product of two trigonometric functions as the sum or difference of two other trigonometric functions.

L

Leibniz's rule: a rule which gives a formula for the differentiation of a definite integral whose limits are functions of the differentiable variables.

lower limit: in a definite integral, the lower limit is the first number in the interval.

M

Mean Value Theorem: a theorem that is used to find a number c in an interval $|a, b|$ such that $f(c)$ is the ratio of the definite integral on the given interval to the difference of a and b.

P

partial fraction: when a complicated fraction is given we can write it as the sum of simpler fractions. These fractions are called partial fractions.

primitive of a function: the antiderivative of a function.

R

radical function: a function which contains one or more radical expressions such square roots, cube roots, etc.

rational function: a function which is written as the quotient of polynomials.

reducible function: a function that can be written in a simpler form, or as the multiplication of simpler functions.

S

sign function: a function that gives the sign of a function $f(x)$. If $f(x)$ is positive then the sign function has value $+1$, if $f(x)$ is 0 then the sign is 0 and when $f(x)$ is negative its signum is -1.

solid of revolution: a solid figure that is generated by rotating any curve or the graph of a function around the x or y axis.

substitution method: the method for finding the integral of a function by using different and suitable variables instead of x.

surface of revolution: the surface area of a solid of revolution.

T

tan x/2 substitution: a method for integrating a function which includes linear expressions of $\sin x$ and $\cos x$ by using the substitution $t = \tan (x/2)$.

trigonometric substitution: a method for finding the integral of a radical expression which includes expressions such as $x^2 \pm a^2$ or $a^2 \pm x^2$.

U

upper limit: in a definite integral, the upper limit is the last number in the interval.

V

volume of revolution: the volume of a solid of revolution.

BASIC DERIVATIVE FORMULAS

- $\frac{d}{dx}(c) = 0$
- $\frac{d}{dx}(x^n) = nx^{n-1}$
- $\frac{d}{dx}(c \cdot f(x)) = c \cdot f'(x)$
- $\frac{d}{dx}[f(x) + g(x)] = f'(x) + g'(x)$
- $\frac{d}{dx}[f(x) - g(x)] = f'(x) - g'(x)$
- $\frac{d}{dx}[f(x) \cdot g(x)] = f'(x) \cdot g(x) + f(x) \cdot g'(x)$
- $\frac{d}{dx}\frac{f(x)}{g(x)} = \frac{f'(x) \cdot g(x) - g'(x) \cdot f(x)}{g^2(x)}$
- $\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}$
- $\frac{d}{dx}(u^n) = n \cdot u^{n-1} \frac{du}{dx}$
- $\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{y'(t)}{x'(t)}$

- $(\sin x)' = \cos x$
- $(\cos x)' = -\sin x$
- $(\tan x)' = \sec^2 x = 1 + \tan^2 x$
- $(\cot x)' = -\csc^2 x = -(1 + \cot^2 x)$
- $(\sec x)' = \sec x \cdot \tan x$
- $(\csc x)' = -\csc x \cdot \cot x$
- $(\arcsin x)' = \frac{1}{\sqrt{1 - x^2}}$
- $(\arccos x)' = -\frac{1}{\sqrt{1 - x^2}}$
- $(\arctan x)' = \frac{1}{1 + x^2}$
- $(\ln x)' = \frac{1}{x}$
- $(\log_a x)' = \frac{1}{x \cdot \ln a}$
- $(e^x)' = e^x$
- $(a^x)' = a^x \cdot \ln a$